

# ITÔ STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS OF HURST PARAMETER

$$H > 1/2$$

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**ABSTRACT.** This paper studies the existence and uniqueness of solution of Itô type stochastic differential equation  $dx(t) = b(t, x(t), \omega)dt + \sigma(t, x(t), \omega)dB(t)$ , where  $B(t)$  is a fractional Brownian motion of Hurst parameter  $H > 1/2$  and  $dB(t)$  is the Itô differential defined by using Wick product or divergence operator. The coefficients  $b$  and  $\sigma$  are random and anticipative. Using the relationship between the Itô and pathwise integrals we first write the equation as a stochastic differential equation involving pathwise integral plus a Malliavin derivative term. To handle this Malliavin derivative term the equation is then further reduced to a system of characteristic equations without Malliavin derivative, which is then solved by a careful analysis of Picard iteration, with a new technique to replace the Grönwall lemma which is no longer applicable. The solution of this system of characteristic equations is then applied to solve the original Itô stochastic differential equation up to a positive random time. In special linear and quasilinear cases the global solutions are proved to exist uniquely.

## 1. INTRODUCTION

Let  $T \in (0, \infty)$  be a given fixed number and let  $\Omega$  be the Banach space of continuous real-valued functions  $f : [0, T] \rightarrow \mathbb{R}$  with the supremum norm:  $\|f\| = \sup_{0 \leq t \leq T} |f(t)|$ . For any  $t \in [0, T]$  define the coordinate mapping  $B(t) : \Omega \rightarrow \mathbb{R}$  by  $B(t)(\omega) = \omega(t)$ . Let  $\mathbb{P} = \mathbb{P}^H$  be the probability measure on the Borel  $\sigma$ -algebra  $\mathcal{F}$  of  $\Omega$  such that  $B = (B(t), 0 \leq t \leq T)$  is a fractional Brownian motion of Hurst parameter  $H \in (0, 1)$ . Namely, on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $B = (B(t), 0 \leq t \leq T)$  is a centered (mean 0) Gaussian process of covariance given by

$$\mathbb{E}(B(t)B(s)) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) .$$

Throughout the paper, we consider the case  $H > 1/2$ . The natural filtration generated by  $B(t)$  is denoted by  $\mathcal{F}_t$ . For any  $\beta \in (0, H)$ , it is known that almost surely,  $B(t)$  is Hölder continuous of exponent  $\beta$ . This means that there is a measurable subset of  $\Omega$  of probability one such that any element  $\omega$  in this set  $B(\cdot, \omega)$  is Hölder continuous of exponent  $\beta$ . We shall work on this subset of  $\Omega$  and with an abuse of notation we shall denote this subset still by  $\Omega$ . We also choose and fix such a  $\beta \in (1/2, H)$  throughout the paper.

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Fractional Brownian motions have been received a great attention in recent years. Stochastic integral, Itô formula, and many other basic results have been established. The stochastic differential equation of the form

$$dx(t) = b(t, x(t))dt + \sigma(t, x(t))\delta B(t), \quad 0 \leq t \leq T, \quad x(0) \text{ is given}, \quad (1.1)$$

has been studied by many authors and has found many applications in various fields, where  $\delta$  denotes the *pathwise type integral* defined by using Riemann sum. Among many references we refer to [1, 9, 22] and in particular the references therein for more details. Lyons and his collaborators' work on rough path analysis is a powerful tool in analyzing this type of equation (see [5, 19] and the references therein).

However, in the case when  $B$  is the Brownian motion, the most studied equation is of Itô type. Namely, in (1.1) Itô stochastic differential is used instead of the pathwise one. There are many reasons for the use of Itô stochastic differential in classical Brownian motion case. One reason is from the modeling point of view. If one uses (1.1) to model the state of a certain system, then the term  $b(t, x(t))$  represents *all* the “mean rate of change” of the system and the term  $\sigma(t, x(t))\delta B(t)$  is the “random perturbation”, which has a zero mean contribution.

When we use stochastic differential equations driven by fractional Brownian motion to model natural or social system, we also wish to separate the two parts: the part  $b(t, x(t))$  represents *all* the mean rate of change and the part  $\sigma(t, x(t))\delta B(t)$  is merely the random perturbation, which should have a mean 0. In another word, it is natural to require the *mean* of  $\sigma(t, x(t))\delta B(t)$  in (1.1) to be zero. On the other hand, it is well-known from the work of [4, 6] that if  $H \neq 1/2$ , then the pathwise type stochastic integral with respect to fractional Brownian motion may not be of zero mean. Namely, it is possible that  $\mathbb{E} \left[ \int_0^T \sigma(t, x(t))\delta B(t) \right] \neq 0$ . Motivated by this phenomenon, an Itô type stochastic integral  $\int_0^T \sigma(t, x(t))dB(t)$  is introduced with the use of Wick product in [4, 13] (see [1, 6] and the references therein). This integral has the property that the expectation  $\mathbb{E} \left[ \int_0^T \sigma(t, x(t))dB(t) \right]$  is always equal to zero. This motivates to replace the pathwise integral in (1.1) by the Itô one. In other words, we are led to consider the following *Itô stochastic differential equation*

$$dx(t) = b(t, x(t))dt + \sigma(t, x(t))dB(t), \quad 0 \leq t \leq T, \quad x(0) \text{ is given}, \quad (1.2)$$

where  $dB(t)$  denotes the *Itô type stochastic differential* (divergence type integral) defined in [4] (see also [1, 6, 13] and references therein),  $b$  and  $\sigma$  are two real-valued functions from  $[0, T] \times \mathbb{R}$  to  $\mathbb{R}$  satisfying some conditions that will be made precise later (we shall allow them to be random). To solve the above equation (1.2), a natural approach to try is the Picard iteration. To explain the difficulty let us define  $x_n(t)$  by the following recursive formula (naive Picard iteration):

$$x_n(t) = x(0) + \int_0^t b(s, x_{n-1}(s))ds + \int_0^t \sigma(s, x_{n-1}(s))dB(s), \quad 0 \leq t \leq T, \quad (1.3)$$

where  $n = 1, 2, \dots$  and  $x_0(t) := x(0)$  for all  $0 \leq t \leq T$ . Consider the above stochastic integral term on the right hand side. An Itô isometry formula states

that

$$\begin{aligned} & \mathbb{E} \left( \int_0^t \sigma(s, x_{n-1}(s)) dB(s) \right)^2 \\ = & \mathbb{E} \left\{ \int_0^t \int_0^t \phi(u, v) \sigma(u, x_{n-1}(u)) \sigma(v, x_{n-1}(v)) du dv \right. \\ & \left. + \int_0^t \int_0^t \sigma_x(u, x_{n-1}(u)) \sigma_x(v, x_{n-1}(v)) \mathbb{D}_v^\phi x_{n-1}(u) \mathbb{D}_u^\phi x_{n-1}(v) du dv \right\}, \end{aligned}$$

where  $\phi(u, v) = H(2H - 1)|u - v|^{2H-2}$ ,  $\sigma_x(t, x)$  denotes the partial derivative of  $\sigma(t, x)$  with respect to  $x$  and  $\mathbb{D}_u^\phi$  is the Malliavin derivative (see forthcoming definition (2.10) in next section). From this identity one sees that to bound the  $L^2$  norm of  $x_n$  one has to use the  $L^2$  norm of  $x_{n-1}$  plus the  $L^2$  norm of the Malliavin derivative of  $x_{n-1}$ . In a similar way to bound the Malliavin derivative one has to use the second order Malliavin derivative, and so on. Thus, we see that the naive Picard iteration approximation cannot be applied to study the Itô stochastic differential equation (1.2).

We shall use a different approach to study (1.2). To explain this approach we first use the relationship between pathwise and Itô stochastic integrals (established for example in [4, Theorem 3.12]. See also [1] and [6]) to write the equation (1.2) as

$$\begin{aligned} x(t) &= x(0) + \int_0^t b(s, x(s)) ds + \int_0^t \sigma(s, x(s)) \delta B(s) \\ &\quad - \int_0^t \sigma_x(s, x(s)) \mathbb{D}_s^\phi x(s) ds, \quad 0 \leq t \leq T. \end{aligned} \quad (1.4)$$

Thus, the equation (1.2) is reduced to an equation involved the pathwise integral plus a Malliavin derivative term. To understand the character of this equation, we consider heuristically the dependence on the random element  $\omega \in \Omega$  of the random variable  $x(t, \omega)$  as a function of infinitely many variables (defined on  $\Omega$ ). We write it formally as  $x(t, \omega) = u(t, \tilde{\ell}_1, \dots, \tilde{\ell}_n, \dots)$ , where  $u(t, x_1, x_2, \dots)$  is a function of infinitely many variable and  $\ell_1, \dots, \ell, \dots$  are smooth deterministic functions such

that  $\langle \ell_i, \ell_j \rangle_{\mathcal{H}_\phi} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{otherwise} \end{cases}$  (see the forthcoming definition (2.9) for the

Hilbert space  $\mathcal{H}_\phi$ . We usually assume that  $\{\ell_1, \dots, \ell, \dots\}$  to be an orthonormal basis of  $\mathcal{H}_\phi$  and  $\tilde{\ell}_i = \int_0^T \ell_i(s) dB(s)$ . Thus  $\mathbb{D}_s^\phi x(s) = \sum_{i=1}^\infty \phi_i(s) \frac{\partial u}{\partial x_i}(s, x_1, x_2, \dots)$  with  $\phi_i(s) = \int_0^T \phi(s, r) \ell_i(r) dr$ . With the above notations, the equation (1.4) can be written as (we omit the explicit dependence of  $u$  on  $(x_1, x_2, \dots)$ )

$$\begin{aligned} u(t) &= u(0) + \int_0^t b(s, u(s)) ds + \int_0^t \sigma(s, u(s)) \delta B(s) \\ &\quad - \sum_{i=1}^\infty \int_0^t \phi_i(s) \sigma_x(s, u(s)) \frac{\partial u}{\partial x_i} ds, \quad 0 \leq t \leq T. \end{aligned} \quad (1.5)$$

This is a first order hyperbolic partial differential equation driven by fractional Brownian motion for a function of infinitely many variables. We shall use the idea of characteristic curve approach from the theory of the first order (finitely many variables) hyperbolic equations (see for example [18, 26, 27]). But since the classical

theory is not directly applicable here we need to find the characteristic curve and prove the existence and uniqueness of the solution. Let us also point out that stochastic hyperbolic equations has also been studied by several authors (see e.g. [17]), which is different from ours.

Now we explain our approach to solve (1.4). First, we construct the following coupled system of characteristic equations:

$$\begin{cases} \Gamma(t) = \omega + \int_0^t \sigma_x(s, z(s)) \int_0^s \phi(s, u) du ds; \\ z(t) = \eta(\omega) + \int_0^t b(s, z(s)) ds + \int_0^t \sigma(s, z(s)) \delta B(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z(s)) \sigma_x(u, z(u)) \phi(s, u) du ds, \end{cases} \quad (1.6)$$

where  $\sigma_x$  denotes the partial derivative with respect to  $x$ ,  $\Gamma(t) : \Omega \rightarrow \Omega$  and  $z(t) : \Omega \rightarrow \mathbb{R}$ . This system of equations comes from the characteristic curve equation for the first order hyperbolic equation (1.5) of a function of infinitely many variables (see [18, 26, 27]). For this system of characteristic equations, we shall show the following statements.

- (i) We use the Picard iteration approach to show that the above system of equations (1.6) has a unique solution. Since the fractional Brownian motion  $B$  is not differentiable, the powerful Grönwall lemma cannot no longer be used. Additional effort is needed to solve the corresponding (1.6). We use a different contraction argument, presented in Section 4. We call this approach *fractional Picard iteration* and hope that this general contraction principle may also be useful in solving other equations involving Hölder continuous controls. Let us point out that (1.6) has a global solution.
- (ii) We show in Section 5 that  $\Gamma(t) : \Omega \rightarrow \Omega$  defined by (1.6) has an inverse  $\Lambda(t)$  when  $t$  is sufficiently small (smaller than a positive random constant) and  $x(t, \omega) = z(t, \Lambda(t, \omega))$  satisfies (1.4) (or (1.2)). To this end we need to use a new Itô formula which is quite interesting itself. This new Itô formula is presented in Section 3.
- (iii) For general nonlinear equation (1.2) we can only solve the equation up to a positive (random) time. This is because the inverse  $\Lambda(t)$  of  $\Gamma(t) : \Omega \rightarrow \Omega$  exists only up to some random time (see one example given in Section 5). However, for linear or quasilinear equation we can solve the equation for all time  $t \geq 0$ . In particular, in one dimensional linear case, we can find the explicit solution. This is done in Section 6.

For notational simplicity we only discuss one dimensional equation. The system of several equations can be handled in a similar way. It is only notationally more complex. On the other hand, our approach works for more general random anticipative coefficients with general anticipative random initial conditions. We present our work in this generality. This means we shall study a slightly more general equation (see (5.4) in Section 5) instead of (1.6). There has been an intensive study on anticipative stochastic differential equations by using anticipative calculus (see [2, 3, 23]). We hope our work can shed some lights to this topic as well. Some preliminary results are presented in Section 2 and some notations used in this paper are also fixed there.

## 2. PRELIMINARY

**2.1. Fractional integrals and derivatives.** Denote  $(-1)^\alpha = e^{i\pi\alpha}$  and  $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ . Let  $a, b \in \mathbb{R}$  with  $a < b$  and let  $f \in L^1(a, b)$  and  $\alpha > 0$ . The left-sided and right-sided fractional Riemann-Liouville integrals of  $f$  are defined by

$$I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

and

$$I_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,$$

respectively if the above integrals exist, where  $a \leq t \leq b$ . The Weyl derivatives are defined as (if the integrals exist)

$$D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) \quad (2.1)$$

and

$$D_{b-}^\alpha f(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right). \quad (2.2)$$

For any  $\lambda \in (0, 1)$ , denote by  $C^\lambda(a, b)$  the space of  $\lambda$ -Hölder continuous functions on the interval  $[a, b]$ . We will make use of the notations

$$\|x\|_{a,b,\beta} = \sup_{a \leq \theta < r \leq b} \frac{|x_r - x_\theta|}{|r - \theta|^\beta},$$

and

$$\|x\|_{a,b} = \sup_{a \leq r \leq b} |x_r|,$$

where  $x : \mathbb{R}^d \rightarrow \mathbb{R}$  is a given continuous function. We refer to [25] for more details on fractional integrals and derivatives.

Let  $\pi : a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$  be a partition of  $[a, b]$  and denote  $|\pi| = \max_{0 \leq i \leq n-1} (t_{i+1} - t_i)$ . Assume that  $f \in C^\lambda(a, b)$  and  $g \in C^\mu(a, b)$  with  $\lambda + \mu > 1$ .

For these two functions, we define the Riemann sum  $S_\pi(f|g) = \sum_{i=0}^{n-1} f(t_i)(g(t_{i+1}) - g(t_i))$ . From a classical result of Young [28], we know that as  $|\pi| \rightarrow 0$ , the limit of  $S_\pi(f|g)$  exists and is called the Riemann-Stieltjes integral

$$\int_a^b f dg = \lim_{|\pi| \rightarrow 0} S_\pi(f|g).$$

We also have the following proposition.

**Proposition 2.1.** *Suppose that  $f \in C^\lambda(a, b)$  and  $g \in C^\mu(a, b)$  with  $\lambda + \mu > 1$ . Let  $1 - \mu < \alpha < \lambda$ . Then the Riemann Stieltjes integral  $\int_a^b f dg$  exists and it can be expressed as*

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt, \quad (2.3)$$

where  $g_{b-}(t) = g(t) - g(b)$ .

We also note that in the convergent Riemann sum, we can also use

$$\tilde{S}_\pi(f|g) = \sum_{i=0}^{n-1} f(\xi_i)(g(t_{i+1}) - g(t_i)),$$

where  $\xi_i$  is any point in  $[t_i, t_{i+1}]$ .

Let  $\Omega, \mathcal{H}$  be two separable Banach spaces such that  $\mathcal{H}$  is continuously embedded in  $\Omega$ . Let  $\mathbb{B}$  be another separable Banach space. A mapping  $F : \Omega \rightarrow \mathbb{B}$  is called  $\mathcal{H}$ -differentiable if there is a bounded linear mapping from  $\mathcal{H}$  to  $\mathbb{B}$  (if such mapping exists, then it is unique and is denoted by  $\mathbb{D}F(\omega)$ ) such that

$$\frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon} = \mathbb{D}F(\omega)(h) + o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0; \quad \forall \omega \in \Omega, h \in \mathcal{H}. \quad (2.4)$$

We can also consider  $\mathbb{D}F(\omega)$  as an element in the tensor product space  $\mathbb{B} \otimes \mathcal{H}'$ , where  $\mathcal{H}'$  is the dual of  $\mathcal{H}$ . The directional derivative  $\mathbb{D}_h F$  for any direction  $h \in \mathcal{H}$  is defined as  $\mathbb{D}_h F(\omega) = \mathbb{D}F(\omega)(h) = \langle \mathbb{D}F(\omega), h \rangle_{\mathcal{H}', \mathcal{H}}$ . To simplify notation we also write  $\mathbb{D}_h F(\omega) = \mathbb{D}F(\omega)h$ . If  $\mathcal{H}$  and  $\mathbb{B}$  are Banach spaces of real functions, and if there is a function  $g(s, \omega)$  such that  $\mathbb{D}_h F(\omega) = \mathbb{D}F(\omega)(h) = \int_0^T f(s, \omega)h(s)ds$ ,  $\forall h \in \mathcal{H}$ , then we denote  $\mathbb{D}_s F(\omega) = g(s, \omega)$ .

It is easy to see that we have the following chain rule. If  $F : \Omega \rightarrow \mathbb{B}_1$  is  $\mathcal{H}_1$ -differentiable and  $G : \mathbb{B}_1 \rightarrow \mathbb{B}_2$  is  $\mathcal{H}_2$ -differentiable such that  $\mathbb{D}F(\omega)$  is a bounded mapping from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ , then  $G \circ F$  is also  $\mathcal{H}_1$ -differentiable and

$$\mathbb{D}G(F(\omega)) = (\mathbb{D}G)(F(\omega)) \circ \mathbb{D}F(\omega). \quad (2.5)$$

For any function  $f(t) = f(t, \omega)$ , where  $(t, \omega) \in [0, T] \times \Omega$ , which is Hölder continuous with respect to  $t$  of exponent  $\mu > 1 - H$ , by Proposition 2.1, we can define the pathwise integral  $\int_0^t f(s)\delta B(s)$  as the (pathwise) limit as  $|\pi| = \max_{0 \leq i \leq n-1} (t_{i+1} - t_i) \rightarrow 0$  of the following Riemann sum

$$S_\pi(f) = \sum_{i=0}^{n-1} f(s_i)(B(s_{i+1}) - B(s_i)), \quad (2.6)$$

where  $\pi : 0 = s_0 < s_1 < \dots < t_{n-1} < t_n = t$  is a partition of  $[0, t]$ . This integral can also be given by

$$\int_0^t f(s)\delta B(s) = \int_0^t D_{0+}^{1-\alpha} f(s) D_{t-}^\alpha B_{t-}(s) ds,$$

where  $\alpha$  satisfies  $1 - \mu < \alpha < H$ .

If  $f$  is Hölder continuous of exponent greater than  $1 - H$  and if  $g$  is continuous, then  $\eta(t) = \eta(0) + \int_0^t f(s)\delta B(s) + \int_0^t g(s)ds$  is well-defined. For any continuous function  $F$  on  $[0, T] \times \mathbb{R}$ , which is continuously differentiable in  $t$  and twice continuously in  $x$  we have the following Itô formula:

$$\begin{aligned} F(t, \eta(t)) &= F(0, \eta(0)) + \int_0^t \left[ \frac{\partial}{\partial s} F(s, \eta(s)) + \frac{\partial}{\partial x} F(s, \eta(s))g(s) \right] ds \\ &\quad + \int_0^t \frac{\partial}{\partial x} F(s, \eta(s))f(s)\delta B(s). \end{aligned} \quad (2.7)$$

[see for example [9] and references therein.] We also notice that if  $f : [0, T] \times \Omega \rightarrow \mathbb{R}$  is Hölder continuous of exponent  $\mu > 1 - H$ , then in the Riemann sum (2.6) the

left point  $s_k$  can be replaced by any points  $\xi_k$  in the subinterval. Namely,

$$\int_0^T f(s) \delta B_s = \lim_{|\pi| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_k) (B(s_{k+1}) - B(s_k)), \quad (2.8)$$

where  $\xi_k$  is any point in  $[s_k, s_{k+1}]$ .

In the stochastic analysis of fractional Brownian motions of Hurst parameter  $H > 1/2$ , usually, we take the above Banach space  $\mathcal{H}$  to be the reproducing kernel Hilbert space  $\mathcal{H}_\phi$ :

$$\mathcal{H}_\phi = \left\{ f : [0, T] \rightarrow \mathbb{R}, \|f\|_{\mathcal{H}_\phi}^2 = \int_0^T \int_0^T f(u) f(v) \phi(u-v) du dv < \infty \right\}, \quad (2.9)$$

which is the completion of the space of smooth functions on  $[0, T]$  with respect to the norm  $\|\cdot\|_{\mathcal{H}_\phi}$ , where

$$\phi(u) := H(2H-1)|u|^{2H-2}.$$

The element in  $\mathcal{H}_\phi$  may be generalized function (distribution) although we still write  $f : [0, T] \rightarrow \mathbb{R}$  in (2.9). We can define  $\mathbb{D}_s F(\omega)$  as usual and we denote

$$\mathbb{D}_t^\phi F(\omega) = \int_0^T \phi(t, s) \mathbb{D}_s F(\omega) ds. \quad (2.10)$$

The expectation  $\mathbb{E} \int_a^b f(s) \delta B(s)$  may generally not be zero. In [4] (see also [1, 6]) we introduce an Itô stochastic integral by using the Wick product. We also established a relationship between pathwise and Itô integrals. Here, we can use this relationship to define Itô integral as

$$\int_0^T f(t) dB(t) = \int_0^T f(t) \delta B(t) - \int_0^T \mathbb{D}_t^\phi f(t) dt \quad (2.11)$$

if  $f$  is Hölder continuous of exponent  $\mu > 1-H$  and  $\mathbb{D}_s^\phi f(s)$  exists and is integrable. It is easy to see that  $\mathbb{E} \left( \int_0^T f(t) dB(t) \right) = 0$ .

The Itô formula and many other results for Itô stochastic integral have been established. Here, we explain that the Itô formula for Itô integral can also be obtained from (2.7) and (2.11).

**Proposition 2.2.** *Let*

$$\eta(t) = \eta + \int_0^t f(s) \delta B_s + \int_0^t g(s) ds,$$

where  $f$  is Hölder continuous with exponent greater than  $1-H$  and  $g$  is continuous. Assume that  $\mathbb{D}_s^\phi f(s)$  exists and is a continuous function of  $s$ . Let  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable in  $t$  and twice continuously differentiable in  $x$ . Then

$$\begin{aligned} F(t, \eta(t)) &= F(0, \eta_0) + \int_0^t \frac{\partial F}{\partial x}(s, \eta(s)) f(s) \delta B(s) \\ &\quad + \int_0^t \left[ \frac{\partial F}{\partial s}(s, \eta(s)) + \frac{\partial F}{\partial x}(s, \eta(s)) g(s) + \frac{\partial^2 F}{\partial x^2}(s, \eta(s)) f(s) \mathbb{D}_s^\phi \eta(s) \right] ds. \end{aligned} \quad (2.12)$$

*Proof* We briefly sketch the proof. First, by (2.11) we see

$$\eta(t) = \eta_0 + \int_0^t f(s) \delta B_s + \int_0^t [g(s) - \mathbb{D}_s^\phi f(s)] ds.$$

From (2.7) it follows that

$$\begin{aligned}
F(t, \eta(t)) &= F(0, \eta_0) + \int_0^t \frac{\partial F}{\partial s}(s, \eta(s)) ds + \int_0^t \frac{\partial F}{\partial x}(s, \eta(s)) f(s) \delta B(s) \\
&\quad + \int_0^t \frac{\partial F}{\partial x}(s, \eta(s)) [g(s) - \mathbb{D}_s^\phi f(s)] ds \\
&= F(0, \eta_0) + \int_0^t \frac{\partial F}{\partial s}(s, \eta(s)) ds + \int_0^t \frac{\partial F}{\partial x}(s, \eta(s)) f(s) dB(s) \\
&\quad + \int_0^t \left\{ \frac{\partial F}{\partial x}(s, \eta(s)) [g(s) - \mathbb{D}_s^\phi f(s)] + \mathbb{D}_s^\phi \left( \frac{\partial F}{\partial x}(s, \eta(s)) f(s) \right) \right\} ds.
\end{aligned}$$

This is simplified to (2.12). ■

### 3. ITÔ FORMULAS

Denote  $\mathcal{T} = [0, T]$ . If  $X(t) = \eta + \int_0^t f(s) ds + \int_0^t g(s) \delta B(s)$  and if  $F$  is a function from  $\mathcal{T} \times \mathbb{R}$  to  $\mathbb{R}$ , then an Itô formula for  $F(t, X(t))$  is given by (2.7), or (2.12) if the integral is Itô type (see also [28, 29, 9] and in particular the references therein). However, to show the existence and uniqueness of the solution to Itô stochastic differential equation we need an Itô formula of the following form: If  $X$  is as above,  $\Gamma : \mathcal{T} \times \Omega \rightarrow \Omega$  and  $F : \mathcal{T} \times \mathbb{R} \times \Omega$  to  $\mathbb{R}$ , we want to find an Itô formula for  $F(t, X(t), \Gamma(t))$ . Here and in what follows, we omit the explicit dependence on  $\omega$  when it is clear.

**Lemma 3.1.** *Let  $h(t, u, \omega), (t, u, \omega) \in \mathcal{T}^2 \times \Omega$  be a continuous function of  $t$  and  $u$ . Define a family of nonlinear transforms from  $\Omega$  to  $\Omega$  by*

$$\Gamma(t, \omega) = \omega + \int_0^t h(t, u, \omega) du, \quad t \in \mathcal{T}. \quad (3.1)$$

*Let  $f : \mathcal{T} \times \Omega \rightarrow \mathbb{R}$  be measurable such that for any  $\omega \in \Omega$ ,  $f : \mathcal{T} \rightarrow \mathbb{R}$  is Hölder continuous of order greater than  $1 - H$  so that  $F = \int_0^T f(s) \delta B(s)$  is well-defined. We have*

$$F \circ \Gamma(t, \omega) = \int_0^T f(s, \Gamma(t, \omega)) \delta B(s) + \int_0^T f(s, \Gamma(t, \omega)) h(t, s, \omega) ds. \quad (3.2)$$

*Proof* By a limiting argument we may assume that  $f$  is of the form

$$f(t, \omega) = \sum_{k=0}^{n-1} a_k(\omega) I_{[t_k, t_{k+1})}(t),$$

where  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$  is a partition of the interval  $[0, T]$ . Thus

$$\begin{aligned}
F(\omega) &= \sum_{k=0}^{n-1} a_k(\omega) [B(t_{k+1}, \omega) - B(t_k, \omega)] \\
&= \sum_{k=0}^{n-1} a_k(\omega) [\omega(t_{k+1}) - \omega(t_k)].
\end{aligned}$$



Thus

$$\begin{aligned}
F(\Gamma(t, \omega)) &= \sum_{k=0}^{n-1} a_k(\Gamma(t, \omega)) \left\{ \omega(t_{k+1}) + \int_0^{t_{k+1}} h(t, s, \omega) ds - \omega(t_k) - \int_0^{t_k} h(t, s, \omega) ds \right\} \\
&= \sum_{k=0}^{n-1} \left\{ a_k(\Gamma(t, \omega)) (\omega(t_{k+1}) - \omega(t_k)) + a_k(\Gamma(t, \omega)) \int_{t_k}^{t_{k+1}} h(t, s, \omega) ds \right\} \\
&= \int_0^T f(s, \Gamma(t, \omega)) \delta B(s) + \int_0^T f(s, \Gamma(t, \omega)) h(t, s, \omega) ds,
\end{aligned}$$

which is (3.2). ■

Let  $\mathcal{H} \subseteq \Omega$  be a Banach space continuously embedded in  $\Omega$ . Now we state our new Itô formula.

**Theorem 3.2.** *Let measurable functions  $\eta : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ ,  $f_0, f_1 : \mathcal{T} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ ,  $g_0, g_1 : \mathcal{T} \times \Omega \rightarrow \mathbb{R}$  satisfy*

$$\begin{cases} f_0(s, x, \omega) \text{ and } g_0(s, \omega) \text{ are continuous in } s \in \mathcal{T}; \\ f_1(s, x, \omega) \text{ and } g_1(s, \omega) \text{ are Hölder continuous} \\ \quad \text{with respect to } s \text{ of order greater than } 1 - H; \\ f_1(s, x, \omega) \text{ is Lipschitz in } x. \end{cases}$$

Define

$$F(t, x, \omega) = \eta(x, \omega) + \int_0^t f_0(s, x, \omega) ds + \int_0^t f_1(s, x, \omega) \delta B(s) \quad (3.3)$$

and

$$G(t, \omega) = \xi(\omega) + \int_0^t g_0(s, \omega) ds + \int_0^t g_1(s, \omega) \delta B(s). \quad (3.4)$$

Assume that  $F$  and  $\frac{\partial}{\partial x} F(t, x, \omega)$  are Hölder continuous in  $t$  of exponent greater than  $1 - H$  and Lipschitz in  $x$ ,  $\mathcal{H}$ -differentiable in  $\omega$ . Let  $\xi : \Omega \rightarrow \mathbb{R}$  be measurable. Let  $h$  and  $\Gamma$  be defined as in Lemma 3.1 and assume that  $\Gamma : [0, T] \times \Omega \rightarrow \Omega$  is continuously differentiable in  $s$  with respect to the topology of  $\mathcal{H}$  (namely,  $\frac{d}{ds} \Gamma(s) \in \mathcal{H}$ ). Then

$$\begin{aligned}
F(t, G(t), \Gamma(t)) &= \eta(\xi(\omega), \omega) + \int_0^t f_0(s, G(s), \Gamma(s)) ds \\
&+ \int_0^t f_1(s, G(s), \Gamma(s)) \delta B(s) + \int_0^t f_1(s, G(s), \Gamma(s)) h(s, s, \omega) ds \\
&+ \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_0(s) ds + \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_1(s) \delta B(s) \\
&+ \int_0^t (\mathbb{D}F)(s, G(s), \Gamma(s)) \frac{d}{ds} \Gamma(s) ds, \tag{3.5}
\end{aligned}$$

where and in what follows we denote

$$(\mathbb{D}F)(s, G(s), \Gamma(s)) \frac{d}{ds} \Gamma(s) := \mathbb{D}F(s, x, \omega) \Big|_{x=G(s), \omega=\Gamma(s)} \frac{d}{ds} \Gamma(s).$$

*Proof* Let  $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$  be a partition of  $[0, t]$  and denote  $|\pi| = \max_{0 \leq k \leq n-1} (t_{k+1} - t_k)$ . We have

$$\begin{aligned} F(t, G(t), \Gamma(t)) - F(0, G(0), \Gamma(0)) \\ &= \sum_{k=0}^{n-1} [F(t_{k+1}, G(t_{k+1}), \Gamma(t_{k+1})) - F(t_k, G(t_k), \Gamma(t_k))] \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{k=0}^{n-1} [F(t_{k+1}, G(t_{k+1}), \Gamma(t_{k+1})) - F(t_k, G(t_{k+1}), \Gamma(t_{k+1}))]; \\ I_2 &= \sum_{k=0}^{n-1} [F(t_k, G(t_{k+1}), \Gamma(t_{k+1})) - F(t_k, G(t_k), \Gamma(t_{k+1}))]; \\ I_3 &= \sum_{k=0}^{n-1} [F(t_k, G(t_k), \Gamma(t_{k+1})) - F(t_k, G(t_k), \Gamma(t_k))]. \end{aligned}$$

Let us first look at  $I_1$ . Using Lemma 3.1, we have

$$\begin{aligned} I_1 &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_0(s, G(t_{k+1}), \Gamma(t_{k+1})) ds \\ &\quad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_1(s, x, \omega) \delta B(s) \Big|_{x=G(t_{k+1}), \omega=\Gamma(t_{k+1})} \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_0(s, G(t_{k+1}), \Gamma(t_{k+1})) ds + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_1(s, G(t_{k+1}), \Gamma(t_{k+1})) \delta B(s) \\ &\quad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_1(s, G(t_{k+1}), \Gamma(t_{k+1})) h(t_{k+1}, s) ds. \end{aligned}$$

From here it is easy to see that

$$\begin{aligned} \lim_{|\pi| \rightarrow 0} I_1 &= \int_0^t f_0(s, G(s), \Gamma(s)) ds + \int_0^t f_1(s, G(s), \Gamma(s)) \delta B(s) \\ &\quad + \int_0^t f_1(s, G(s), \Gamma(s)) h(s, s, \omega) ds. \end{aligned} \quad (3.6)$$

Using the mean value theorem we have, denoting  $G_{k,\theta} = G(t_k) + \theta [G(t_{k+1}) - G(t_k)]$ ,

$$\begin{aligned} I_2 &= \sum_{k=0}^{n-1} \int_0^1 \frac{\partial}{\partial x} F(t_k, G_{k,\theta}, \Gamma(t_{k+1})) d\theta [G(t_{k+1}) - G(t_k)] \\ &= \int_0^1 d\theta \left\{ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\partial}{\partial x} F(t_k, G_{k,\theta}, \Gamma(t_{k+1})) [g_0(s) ds + g_1(s) \delta B(s)] \right\}. \end{aligned}$$

Since for any  $\theta \in [0, 1]$ ,  $G_{k,\theta}$  is any point between  $G(t_k)$  and  $G(t_{k+1})$ , we see that for any  $\theta \in [0, 1]$ ,

$$\begin{aligned} & \lim_{|\pi| \rightarrow 0} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\partial}{\partial x} F(t_k, G_{k,\theta}, \Gamma(t_{k+1})) [g_0(s)ds + g_1(s)\delta B(s)] \\ &= \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_0(s)ds + \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_1(s)\delta B(s). \end{aligned}$$

This implies

$$\lim_{|\pi| \rightarrow 0} I_2 = \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_0(s)ds + \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_1(s)\delta B(s). \quad (3.7)$$

$I_3$  can be computed as follows.

$$I_3 = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{D}F(t_k, G(t_k), \Gamma(s)) \frac{d}{ds} \Gamma(s) ds.$$

From here it follow easily

$$\lim_{|\pi| \rightarrow 0} I_3 = \int_0^t \mathbb{D}F(s, G(s), \Gamma(s)) \frac{d}{ds} \Gamma(s) ds. \quad (3.8)$$

Now we combine (3.6), (3.7) and (3.8) to prove the theorem. ■

**Remark 3.3.** (i)  $F(0, G(0), \Gamma(0)) = \eta(\xi(\omega), \omega)$ . Thus, we can replace  $\eta(\xi(\omega), \omega)$  in (3.5) by  $F(0, G(0), \Gamma(0))$ .

(ii) We may also write the above Itô formula (3.5) in differential form as follows.

$$\begin{aligned} dF(t, G(t), \Gamma(t)) &= f_0(t, G(t), \Gamma(t))dt + f_1(t, G(t), \Gamma(t))\delta B(t) \\ &+ \frac{\partial}{\partial x} F(t, G(t), \Gamma(t)) [g_0(t)dt + g_1(t)\delta B(t)] \\ &+ f_1(t, G(t), \Gamma(t))h(t, t, \omega)dt + (\mathbb{D}F)(t, G(t), \Gamma(t)) \frac{\partial}{\partial t} \Gamma(t)dt. \end{aligned} \quad (3.9)$$

If  $F$  and  $G$  are given by Itô integrals, then what will be the Itô formula? Similar to the argument in the proof of Proposition 2.2 we can use the relationship (2.11) to obtain an analogous Itô formula for Itô integrals.

Let

$$F(t, x, \omega) = \eta(x, \omega) + \int_0^t f_0(s, x, \omega)ds + \int_0^t f_1(s, x, \omega)\delta B(s) \quad (3.10)$$

$$G(t, \omega) = \xi(\omega) + \int_0^t g_0(s, \omega)ds + \int_0^t g_1(s, \omega)\delta B(s). \quad (3.11)$$

Let  $h$  and  $\Gamma$  be defined as in Lemma 3.1. We assume the conditions in Theorem 3.2 hold. Moreover, we also assume that  $\mathbb{D}_s f_1^\phi(s)$  and  $\mathbb{D}_s g_1^\phi(s)$  are continuously in  $s$ . From (2.11) we have (we omit the explicit dependence on  $\omega$ )

$$\begin{aligned} F(t, x) &= \eta(x) + \int_0^t [f_0(s, x) - \mathbb{D}_s^\phi f_1(s, x)] ds + \int_0^t f_1(s, x)\delta B(s) \\ G(t) &= \xi + \int_0^t [g_0(s) - \mathbb{D}_s^\phi g_1(s)] ds + \int_0^t g_1(s)\delta B(s). \end{aligned}$$

By the Itô formula (3.5) we have

$$\begin{aligned}
F(t, G(t), \Gamma(t)) &= \eta(\xi) + \int_0^t [f_0 - \mathbb{D}_s^\phi f_1](s, G(s), \Gamma(s)) ds \\
&+ \int_0^t f_1(s, G(s), \Gamma(s)) \delta B(s) + \int_0^t f_1(s, G(s), \Gamma(s)) h(s, s, \omega) ds \\
&+ \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) [g_0 - (\mathbb{D}_s^\phi g_1)](s, \Gamma(s)) ds \\
&+ \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_1(s, \Gamma(s)) \delta B(s) \\
&+ \int_0^t (\mathbb{D}F)(s, G(s), \Gamma(s)) \frac{d}{ds} \Gamma(s) ds.
\end{aligned}$$

Using again the relationship (2.11) between pathwise and Itô integral, we can rewrite the above identity as

$$\begin{aligned}
&F(t, G(t), \Gamma(t)) \\
&= \eta(\xi) + \int_0^t \{ \mathbb{D}_s^\phi [f_1(s, G(s), \Gamma(s))] + [f_0 - \mathbb{D}_s^\phi f_1](s, G(s), \Gamma(s)) \} ds \\
&+ \int_0^t f_1(s, G(s), \Gamma(s)) dB(s) + \int_0^t f_1(s, G(s), \Gamma(s)) h(s, s, \omega) ds \\
&+ \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) [g_0 - (\mathbb{D}_s^\phi g_1)](s, \Gamma(s)) ds \\
&+ \int_0^t \mathbb{D}_s^\phi \left[ \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_1(s, \Gamma(s)) \right] ds \\
&+ \int_0^t \frac{\partial}{\partial x} F(s, G(s), \Gamma(s)) g_1(s, \Gamma(s)) dB(s) \\
&+ \int_0^t (\mathbb{D}F)(s, G(s), \Gamma(s)) \frac{d}{ds} \Gamma(s) ds. \tag{3.12}
\end{aligned}$$

#### 4. AN ITERATION PRINCIPLE

After transforming the original equation (1.2) into the system of equations (1.6) (or (5.4) in next section for general random coefficient case), we can now use the Picard iteration method to solve the new differential system (pathwise). But the second equation in (1.6) involves  $\int_0^t \sigma(s, z(s), \Gamma(s)) \delta B(s)$ . Since  $B(s)$  is not differentiable, one cannot no longer use the powerful Grönwall lemma. One way to get around this difficulty is to use the Besov spaces (see e.g. [24]). Here, we propose to use the Hölder spaces which seems to be simpler. The idea is motivated by the works [11, 12].

In this section we present a general contraction principle, which may be useful in solving other equations driven by Hölder continuous functions. We call this approach the fractional Picard iteration. In next section we shall use this general contraction principle to solve (1.6) and subsequently to solve (1.2).

Let  $\mathbb{B}$  be a separable Banach space with norm  $\|\cdot\|$  (in case we need to specify we write  $\|\cdot\|_{\mathbb{B}}$ ). We denote by  $\mathbb{B}[0, T]$  the Banach space of all continuous functions from  $[0, T]$  to  $\mathbb{B}$  with the sup norm  $\|x\|_{0,T} = \sup_{0 \leq t \leq T} \|x(t)\|_{\mathbb{B}}$ . For any  $0 \leq a < b \leq T$

and an element  $x \in \mathbb{B}[0, T]$ , we define

$$\|x\|_{a,b,\beta} = \sup_{a \leq s < t \leq b} \frac{\|x(t) - x(s)\|}{|t - s|^\beta}$$

if the above right hand side is finite. We also use the notation  $\|x\|_{a,b} = \sup_{a \leq t \leq b} \|x(t)\|$  and in the case when  $a = b$ , we denote  $\|x\|_{a,a} = \|x(a)\|$ .

Given  $\Delta \in (0, T]$  and  $\beta \in (0, 1]$  we denote

$$\|x\|_{\Delta,\beta} = \sup_{0 \leq t \leq T} |x(t)| + \sup_{0 \leq s < t \leq T, t-s \leq \Delta} \frac{\|x(t) - x(s)\|}{|t - s|^\beta}.$$

Denote

$$\mathbb{B}^{\Delta,\beta}[0, T] = \{x \in \mathbb{B}[0, T]; \|x\|_{\Delta,\beta} < \infty\}.$$

As in Theorem 1.3.3 of [16], it is easy to verify that  $\|x\|_{\Delta,\beta}$  is a norm and  $\mathbb{B}^{\Delta,\beta}[0, T]$  is a Banach space with respect to this norm. When we need to emphasize the interval we may also add the interval into the notation, namely, we may write  $\|x\|_{a,b,\Delta,\beta}$ . If  $\Delta$  is clear, we omit the dependence on  $\Delta$  and write  $\mathbb{B}^\beta[0, T] = \mathbb{B}^{\Delta,\beta}[0, T]$ .

We shall consider a mapping  $F$  from  $\mathbb{B}[0, T]$  into itself. Thus, for any element  $x \in \mathbb{B}[0, T]$ ,  $F(x)$  is a function from  $[0, T]$  to  $\mathbb{B}$ . We can thus write such function as  $F(t, x)$ . We say  $F$  is progressive if for any  $a \in [0, T]$ ,  $\{F(t, x), 0 \leq t \leq a\}$  depends only on  $\{x(t), 0 \leq t \leq a\}$ . In other words, if  $x(t) = y(t)$  for all  $t \in [0, a]$ , then  $F(t, x) = F(t, y)$  for all  $t \in [0, a]$ .

Here is the main theorem of this section.

**Theorem 4.1.** *Let  $\mathbb{B}$  be a separable Banach space and let  $F$  be a progressive mapping from  $\mathbb{B}[0, T] \rightarrow \mathbb{B}[0, T]$  such that  $F(0, x) \in \mathbb{B}$  is independent of  $x$  (it is equivalent to say that  $F(0, x) \in \mathbb{B}$  is independent of  $x(0)$ ). Suppose that there are constants  $\kappa, \Delta > 0, \gamma, \beta \in (0, 1]$  and there is a positive function  $h : \mathbb{R}^4 \rightarrow \mathbb{R}$ , increasing in all of its arguments, such that the following statements are true.*

(i) *For any  $0 \leq a < b \leq T$  with  $b - a \leq \Delta$  and for any  $x \in \mathbb{B}^{\Delta,\beta}[0, T]$  we have*

$$\|F(x)\|_{a,b,\beta} \leq \kappa(1 + \|x\|_{0,a} + \|x\|_{a,b,\beta}(b-a)^\gamma). \quad (4.1)$$

(ii) *For any  $0 \leq a < b \leq T$  with  $b - a \leq \Delta$  and for any  $x_1, x_2 \in \mathbb{B}^{\Delta,\beta}[0, T]$  we have*

$$\begin{aligned} \|F(x_1) - F(x_2)\|_{a,b,\beta} &\leq \bar{h}_{a,b}(x_1, x_2) \left\{ \|x_1 - x_2\|_{0,a} \right. \\ &\quad \left. + \|x_1 - x_2\|_{a,b,\beta}(b-a)^\gamma \right\}, \end{aligned} \quad (4.2)$$

where

$$\bar{h}_{a,b}(x_1, x_2) = h(\|x_1\|_{0,a}, \|x_2\|_{0,a}, \|x_1\|_{a,b,\beta}, \|x_2\|_{a,b,\beta}). \quad (4.3)$$

Then the mapping  $F : \mathbb{B}^\beta[0, T] \rightarrow \mathbb{B}^\beta[0, T]$  has a unique fixed point  $x \in \mathbb{B}[0, T]$ . This means that there is a unique  $x \in \mathbb{B}[0, T]$  such that  $x(t) = F(t, x)$  for all  $t \in [0, T]$ . Moreover, there is a  $\tau_0 > 0$  such that

$$\begin{cases} \|x\|_{0,T} \leq c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|), \\ \sup_{0 \leq a < b \leq T, b-a \leq \tau_0} \|x\|_{a,b,\beta} \leq c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|), \end{cases} \quad (4.4)$$

where  $c_1$  and  $c_2$  are two constants depending only on  $\Delta$ .

*Proof* We divide the proof into several steps.

*Step 1.* First, we prove that there is a  $\tau_1 \in (0, \Delta]$  (the choice of  $\tau_1$  will be made more precise later) such that  $F$  has a unique fixed point on the interval  $[0, \tau_1]$ . To this end we use Picard iteration. We define  $x_0(t) = F(0, x)$  for all  $t \in [0, \tau_1]$  which is an element in  $\mathbb{B}$  by our assumption that  $F(0, x)$  is independent of  $x$ . We also define for  $n = 0, 1, 2, \dots$

$$x_{n+1}(t) = F(t, x_n), \quad t \in [0, \tau_1]. \quad (4.5)$$

It is easy to see by the assumption that  $x_n(0) = F(0, x)$  for all  $n \geq 0$ . From the assumption (4.1), we have

$$\begin{aligned} \|x_{n+1}\|_{0, \tau_1, \beta} &\leq \kappa(1 + \|x_n(0)\| + \|x_n\|_{0, \tau_1, \beta} \tau_1^\gamma) \\ &\leq \kappa(1 + \|F(0)\| + \|x_n\|_{0, \tau_1, \beta} \tau_1^\gamma). \end{aligned}$$

Let

$$\tau_1 \leq \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta. \quad (4.6)$$

[One can take  $\tau_1 \leq \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta$ .] Then we have

$$\|x_{n+1}\|_{0, \tau_1, \beta} \leq \kappa(1 + \|F(0, x)\|) + \frac{1}{2} \|x_n\|_{0, \tau_1, \beta}. \quad (4.7)$$

By induction, we have

$$\sup_{n \geq 0} \|x_n\|_{0, \tau_1, \beta} \leq 2\kappa(1 + \|F(0)\|). \quad (4.8)$$

Now by the fact that  $\|x_n\|_{0, \tau_1} \leq \|x_n(0)\| + \|x_n\|_{0, \tau_1, \beta} \tau_1^\gamma$ , we see that

$$\sup_{n \geq 0} \|x_n\|_{0, \tau_1} \leq \|F(0)\| + 2\kappa(1 + \|F(0)\|) \tau_1^\gamma \leq 2(1 + \|F(0)\|). \quad (4.9)$$

By the definition (4.3) of  $\bar{h}$  we see that

$$\sup_n \bar{h}_{0, \tau_1}(x_{n-1}, x_n) \leq M_1 < \infty$$

for some positive constant  $M_1 \in (0, \infty)$ . Notice that  $x_n(0) = x_{n-1}(0)$ . Thus condition (4.2) gives

$$\|F(x_{n+1}) - F(x_n)\|_{0, \tau_1, \beta} \leq M_1 \|x_{n+1} - x_n\|_{0, \tau_1, \beta} \tau_1^\gamma.$$

Choose

$$\tau_1 \leq \frac{1}{(2M_1)^{1/\gamma}} \wedge \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta. \quad (4.10)$$

Then we have

$$\begin{aligned} \|x_{n+1} - x_n\|_{0, \tau_1, \beta} &= \|F(x_n) - F(x_{n-1})\|_{0, \tau_1, \beta} \\ &\leq \frac{1}{2} \|x_n - x_{n-1}\|_{0, \tau_1, \beta}. \end{aligned}$$

Since  $x_n(0) = F(0)$  for all  $n$  this means that  $\{x_n\}$  is a Cauchy sequence in  $\mathbb{B}^\beta[0, \tau_1]$  and it converges to an element  $x \in \mathbb{B}^\beta[0, \tau_1]$ . Obviously, this limit  $x$  is the unique solution to  $x(t) = F(t, x)$  for  $t \in [0, \tau_1]$ . Clearly, the limit satisfies

$$\sup_{n \geq 0} \|x\|_{0, \tau_1} \leq 2(1 + \|F(0)\|), \quad \|x\|_{0, \tau_1, \beta} \leq 2\kappa(1 + \|F(0)\|). \quad (4.11)$$

*Step 2.* Now we explain the inductive argument to construct a unique solution on the interval  $[0, T \wedge T_{k+1}]$  from a solution on  $[0, T \wedge T_k]$ , where  $T_k = \tau_1 + \dots + \tau_k$ . For any positive integer  $k \geq 1$  assume that there is a unique solution  $x(t), t \in$

$[0, T_k]$  satisfying  $x(t) = F(t, x), t \in [0, T_k]$ . We want to construct a unique solution  $x(t), t \in [0, T_{k+1}]$  satisfying  $x(t) = F(t, x), t \in [0, T_{k+1}]$  for some  $\tau_{k+1} > 0$  (see below for the definition of  $\tau_{k+1}$ ). To simplify notation we assume  $T_{k+1} \leq T$  (or we replace  $T_{k+1}$  by  $T_{k+1} \wedge T$ ). Define the following sequence (still use  $x_n$ )

$$\begin{cases} x_0(t) = \begin{cases} x(t), & \text{when } 0 \leq t \leq T_k, \\ x(T_k), & \text{when } T_k \leq t \leq T_{k+1}, \end{cases} \\ x_{n+1}(t) = F(t, x_n), & \text{for all } 0 \leq t \leq T_{k+1}, \end{cases}$$

where  $n = 0, 1, \dots$ . Since  $x(t), t \in [0, T_k]$  is the unique solution to  $x(t) = F(t, x), t \in [0, T_k]$ , we see that  $x_n(t) = x(t)$  for all  $t \in [0, T_k]$ . With exactly the same argument as for (4.7), we have for any positive integer  $k \geq 1$ ,

$$\|x_{n+1}\|_{T_k, T_{k+1}, \beta} \leq \kappa(1 + \|x\|_{0, T_k}) + \frac{1}{2}\|x_n\|_{T_k, T_{k+1}, \beta} \quad (4.12)$$

under the condition

$$\tau_{k+1} \leq \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta. \quad (4.13)$$

[We can take  $\tau_{k+1} = \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta$ ]. This can be used (by induction on  $n$ ) to prove

$$\sup_{n \geq 0} \|x_n\|_{T_k, T_{k+1}, \beta} \leq 2\kappa(1 + \|x\|_{0, T_k}) =: M_{k+1}^{(1)}. \quad (4.14)$$

As a consequence, we have

$$\begin{aligned} \sup_{n \geq 0} \|x_n\|_{0, T_{k+1}} &\leq \|x\|_{0, T_k} + \|x_n\|_{T_k, T_{k+1}, \beta} \tau_{k+1}^\gamma \\ &\leq (2\kappa \tau_{k+1}^\beta + 1)(1 + \|x\|_{0, T_k}) \\ &\leq 2(1 + \|x\|_{0, T_k}) =: M_{k+1}^{(2)}. \end{aligned} \quad (4.15)$$

Now letting  $M_{k+1} := h(M_{k+1}^{(2)}, M_{k+1}^{(2)}, M_{k+1}^{(1)}, M_{k+1}^{(1)})$ , we have by (4.2)

$$\begin{aligned} \|x_{n+1} - x_n\|_{T_k, T_{k+1}, \beta} &= \|F(x_n) - F(x_{n-1})\|_{T_k, T_{k+1}, \beta} \\ &\leq M_{k+1} \|x_n - x_{n-1}\|_{T_k, T_{k+1}, \beta} \tau_{k+1}^\gamma \\ &\leq \frac{1}{2} \|x_n - x_{n-1}\|_{T_k, T_{k+1}, \beta} \end{aligned} \quad (4.16)$$

if

$$\tau_{k+1} \leq \frac{1}{M_{k+1}^{1/\gamma}} \wedge \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta. \quad (4.17)$$

Thus, under the above condition 4.17,  $\{x_n\}$  is a Cauchy sequence in  $\mathbb{B}[0, T_{k+1}]$ . It has a unique limit  $x$  which satisfies  $x(t) = F(t, x)$  for all  $t \in [0, T \wedge T_{k+1}]$ . Indeed, the fact  $x(t)$  satisfies  $x(t) = F(t, x)$  for all  $t \in [0, T \wedge T_k]$  follows from the inductive assumption. On  $[T \wedge T_k, T \wedge T_{k+1}]$ ,  $x_n$  is a Cauchy sequence in  $B^\beta[T \wedge T_k, T \wedge T_{k+1}]$  and  $F$  is continuous on  $B^\beta[T \wedge T_k, T \wedge T_{k+1}]$  by the assumption (4.2). It is also easy to verify from (4.14) and (4.15) that the solution satisfies

$$\begin{cases} \|x\|_{T_k, T_{k+1}, \beta} \leq 2\kappa(1 + \|x\|_{0, T_k}); \\ \|x\|_{0, T_{k+1}} \leq 2(1 + \|x\|_{0, T_k}). \end{cases} \quad (4.18)$$

*Step 3.* Denote  $T_\infty = T \wedge (\tau_1 + \tau_2 + \dots)$ . By induction argument, we can construct a unique solution  $x(t)$  on  $t \in [0, T_\infty]$  for the equation  $x(t) = F(t, x)$ . We want to

show  $T_\infty = T$ . To do this, the idea is to show that  $\tau_k \geq \tilde{\tau}_0$  for some  $\tilde{\tau}_0 > 0$  and for all  $k \geq 1$ .

From the same argument as for (4.14) and (4.15) we have

$$\begin{cases} \|x\|_{k\tau, (k+1)\tau, \beta} \leq 2\kappa(1 + \|x\|_{0, k\tau}); \\ \|x\|_{0, (k+1)\tau} \leq 2(1 + \|x\|_{0, k\tau}) \end{cases} \quad (4.19)$$

as long as  $\tau \leq \frac{1}{(2\kappa)^{1/\gamma}}$  and  $(k+1)\tau \leq T_\infty$ . In fact, to obtain the above bounds, we only need to use the condition (4.1).

We choose  $\tau = \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta$  and divide the interval  $[0, T_\infty]$  into  $N$  sub-intervals, where

$$N = \left\lceil \frac{T_\infty}{\tau} \right\rceil + 1 = \left\lceil \frac{T_\infty}{\Delta} \right\rceil \vee \left\lceil T_\infty (2\kappa)^{1/\gamma} \right\rceil + 1.$$

Denote  $A_k = \|x\|_{0, k\tau}$ . The second inequality in (4.19) can be written as

$$A_{k+1} \leq 2 + 2A_k, \quad k = 0, 1, 2, \dots$$

An elementary induction argument yields

$$\begin{aligned} A_k &\leq 2 + 2^2 + \dots + 2^k + 2^k A_0 \\ &\leq 2^{k+1} + 2^k A_0. \end{aligned}$$

Thus, we see that

$$\begin{aligned} \|x\|_{0, T_\infty} &\leq 2^{N+1} + 2^N \|F(0)\| \\ &\leq c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|) \end{aligned} \quad (4.20)$$

for some constants  $c_1$  and  $c_2$  dependent only on  $\Delta$ . This together with the first inequality in (4.19) yields

$$\|x\|_{k\tau, (k+1)\tau, \beta} \leq c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|) \quad (4.21)$$

for any  $k$  such that  $(k+1)\tau \leq T_\infty$  and for  $\tau = \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta$ .

*Step 4.* Denote

$$\begin{cases} \tilde{M}_1 &= \tilde{M}_2 = c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|); \\ \tilde{M} &= h(\tilde{M}_2, \tilde{M}_2, \tilde{M}_1, \tilde{M}_1). \end{cases}$$

Then from (4.14) and (4.15) we see that  $M_{k+1}^{(1)} \leq \tilde{M}_1$  and  $M_{k+1}^{(2)} \leq \tilde{M}_2$  for all  $k$ . Since  $h$  is increasing in all of its arguments, this means that we can choose  $\tau_k$  such that

$$\tau_k \geq \tilde{\tau} := \frac{1}{\tilde{M}^{1/\gamma}} \wedge \frac{1}{(2\kappa)^{1/\gamma}} \wedge \Delta, \quad \forall k \geq 1.$$

Since  $\tilde{\tau}$  is independent of  $k$ , we see that  $T_\infty = T$ .

The first inequality (4.4) is a straightforward consequence of (4.20). With possibly a different choice of  $c_1$  and  $c_2$ , we can write (4.21) as

$$\|x\|_{k\tau_0, (k+1)\tau_0, \beta} \leq c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|).$$



If  $a, b \in [k\tau_0, (k+1)\tau_0]$ , then we see easily that the second inequality in (4.4) holds. If  $a \in [(k-1)\tau_0, k\tau_0]$  and  $b \in [k\tau_0, (k+1)\tau_0]$ , then

$$\begin{aligned} \frac{\|x(b) - x(a)\|}{|b - a|^\beta} &\leq \frac{\|x(b) - x(k\tau_0)\| + \|x(k\tau) - x(a)\|}{|b - a|^\beta} \\ &\leq \frac{\|x(b) - x(k\tau_0)\|}{|b - k\tau_0|^\beta} + \frac{\|x(k\tau) - x(a)\|}{|k\tau_0 - a|^\beta} \\ &\leq \|x\|_{(k-1)\tau_0, k\tau_0, \beta} + \|x\|_{k\tau_0, (k+1)\tau_0, \beta} \\ &\leq 2c_2 e^{c_1 \kappa^{1/\gamma} T} (1 + \|F(0)\|). \end{aligned}$$

Up to a different choice of constant  $c_2$ , we prove the second inequality in (4.4). ■

**Remark 4.2.** If  $\bar{h}$  has some particular form, one may obtain some stability results for the solutions. Namely, if  $x_1$  and  $x_2$  are two solutions with different initial conditions, or with different  $F$ , one may bound  $\|x_2 - x_1\|$  (see analogous results [11, 12]). However, we shall not persuade this problem.

## 5. GENERAL STOCHASTIC DIFFERENTIAL EQUATIONS

**5.1. Reduction of the equation.** Let  $b, \sigma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  be a measurable mapping. We shall specify the conditions on them later. Let  $\eta$  be a given random variable. The main objective of this paper is to study the following Itô stochastic differential equation

$$\begin{cases} dx(t) = b(t, x(t), \omega)dt + \sigma(t, x(t), \omega)dB(t), & 0 \leq t \leq T, \\ x(0) = \eta, \end{cases} \quad (5.1)$$

where  $dB(t)$  is the Itô differential.

We can use the argument as in the introduction (see e.g. (1.6)) to reduce the above equation (5.1), now with random coefficients. Using the relationship (2.11) between the Itô and pathwise stochastic integrals and the chain rule for derivative we have

$$\begin{aligned} \int_0^t \sigma(s, x(s), \omega)dB(s) &= \int_0^t \sigma(s, x(s), \omega)\delta B(s) - \int_0^t \mathbb{D}_s^\phi [\sigma(s, x(s), \omega)] ds \\ &= \int_0^t \sigma(s, x(s), \omega)\delta B(s) - \int_0^t \mathbb{D}_s^\phi [\sigma](s, x(s), \omega)ds \\ &\quad - \int_0^t \sigma_x(s, x(s), \omega)\mathbb{D}_s^\phi x(s)ds, \end{aligned}$$

where  $\sigma_x$  denotes the partial derivative of  $\sigma$  with respect to  $x$ , and  $\mathbb{D}_s^\phi [\sigma]$  denotes the partial derivative of  $\sigma$  with respect to the random element  $\omega$ . Thus, the equation (5.1) may be written as

$$\begin{aligned} x(t) &= \eta + \int_0^t \tilde{b}(s, x(s), \omega)ds + \int_0^t \sigma(s, x(s), \omega)\delta B(s) \\ &\quad - \int_0^t \tilde{\sigma}(s, x(s), \omega)\mathbb{D}_s^\phi x(s)ds, \end{aligned} \quad (5.2)$$

where

$$\begin{cases} \tilde{b}(s, x, \omega) := b(s, x, \omega) - \mathbb{D}_s^\phi [\sigma](s, x, \omega) \\ \tilde{\sigma}(s, x, \omega) = \sigma_x(s, x, \omega). \end{cases} \quad (5.3)$$

As explained in the introduction, this equation can be considered as a first order nonlinear hyperbolic equation of infinitely many variables, driven by fractional Brownian motion, where  $\omega \in \Omega$  is considered as an infinite dimensional variable. We shall use the elementary characteristic curve method. The characteristic equation will be an equation in  $\Omega$  which takes the form of the first equation of the following system of equations. This means that to solve the above equation (5.2) we will first solve the following coupled system of equations (which we call it the system of characteristic equations corresponding to (5.1)).

$$\begin{cases} \Gamma(t) = \omega + \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s)) \int_0^s \phi(s, u) du ds; \\ z(t) = \eta(\omega) + \int_0^t \tilde{b}(s, z(s), \Gamma(s)) ds + \int_0^t \sigma(s, z(s), \Gamma(s)) \delta B(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z(s), \Gamma(s)) \tilde{\sigma}(u, z(u), \Gamma(u)) \phi(s, u) du ds. \end{cases} \quad (5.4)$$

We shall show that the solution to equation (5.4) can be used to express the solution of (5.2). However, first we need to show that (5.4) has a (unique) solution.

**5.2. Solution to the reduced equation.** In this section we prove that the system (5.4) has a unique solution. When the Hurst parameter  $H > 1/2$  and in the absence of  $\mathbb{D}_s^\phi x(s)$ , the equation (5.2) has been studied by many authors (see [24, 29, 9] for a recent study and also for some more references). We only mention two works. In [24], Besov spaces are used to accommodate the solutions. In [11], the solution is shown to be Hölder continuous and the stability with respect to Hölder norm is also studied in that paper (see [12] for a similar study when the Hurst parameter  $H \in (1/3, 1/2]$ ). Here, we shall use the Hölder spaces together with the general contraction principle established in Section 4 to prove the existence and uniqueness of the solution. Our idea to solve the equation (5.2) seems also new even in the classical case (namely, in the absence of  $\mathbb{D}_s^\phi x(s)$  in (5.2)).

The system of (two) equations (5.4) will be solved for any fixed  $\omega \in \Omega$ . This means that we are going to find pathwise solution of (5.4) by using Theorem 4.1. To this end, we rewrite the equation (5.4) with a replacement of  $\Gamma$  of by  $\Gamma + \omega$  (we use the same notation  $\Gamma(t)$  without ambiguity).

$$\begin{cases} \Gamma(t) = \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s) + \omega) \int_0^s \phi(s, u) du ds; \\ z(t) = \eta(\omega) + \int_0^t \tilde{b}(s, z(s), \Gamma(s) + \omega) ds + \int_0^t \sigma(s, z(s), \Gamma(s) + \omega) \delta B(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z(s), \Gamma(s) + \omega) \tilde{\sigma}(u, z(u), \Gamma(u) + \omega) \phi(s, u) du ds. \end{cases} \quad (5.5)$$

Before we solve (5.5), we need to explain the space that the solution stay. To find such a space to accommodate the above  $\Gamma$  we introduce the following Banach space:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_p = \left\{ h : \mathcal{T} \rightarrow \mathbb{R}; h \text{ is absolutely continuous} \right. \\ &\quad \left. \text{such that } \int_0^T |\dot{h}(s)|^p ds < \infty \right\}, \end{aligned} \quad (5.6)$$

where  $p$  is a number such that  $p \in (1, \frac{1}{2-2H})$  (we shall fix such a number throughout the remaining part of this paper), and where the norm is defined by

$$\|h\|_{\mathcal{H}} = \|h\|_{\mathcal{H}_p} = \left( \int_0^T |\dot{h}(s)|^p ds \right)^{1/p}.$$

It is straightforward to see that any  $h \in \mathcal{H}$  is an element of  $\Omega$  and

$$\|h\|_{\Omega} \leq c_{p,T} \|h\|_{\mathcal{H}}.$$

**Remark 5.1.** The principle to choose the Banach space  $\mathcal{H}$  is as follows. First, we need that  $\frac{d}{dt}\Gamma(t) \in \mathcal{H}$ . Secondly, we want the norm of  $\mathcal{H}$  is as strong as possible so that the coefficients  $\sigma$ ,  $\tilde{\sigma}$ , and  $\tilde{b}$  are differentiable on  $\omega$  with respect to this norm  $\|\cdot\|_{\mathcal{H}}$ . Namely, with respect to  $\omega$ , the coefficients  $\sigma$  and  $b$  and the initial condition  $\eta$  satisfy

$$|\sigma(\omega + h) - \sigma(\omega)| \leq C \|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H}. \quad (5.7)$$

(Similar inequality for  $\tilde{b}$ ,  $\tilde{\sigma}$ , and the initial conditions). Of course, the larger the norm of  $\mathcal{H}$ , the broader the condition (5.7) will be. In the analysis of nonlinear Wiener functionals, it is known that many interesting random variables do not satisfy (5.7) with  $\mathcal{H} = \Omega$  (see the example of Lévy area in [10]). But usually (5.7) is satisfied when  $\mathcal{H}$  is the Cameron-Martin norm, which is given by  $\|h\|_{\mathcal{H}_\phi}^2 := \int_0^T \int_0^T \phi(u-v)h(u)h(v)dudv$ , in our case of fractional Brownian motion. Namely, for many random variables in stochastic analysis, such as the solution of a stochastic differential equation, we have

$$|f(\omega + h) - f(\omega)| \leq C \|h\|_{\mathcal{H}_\phi}, \quad \forall h \in \mathcal{H}_\phi. \quad (5.8)$$

An inequality of Littlewood-Paley type ([20]) states

$$\|h\|_{\mathcal{H}_\phi} \leq C_H \|h\|_{\mathcal{H}_q}, \quad \forall h \in \mathcal{H}_q, \quad \text{with } q := 1/H.$$

When  $H > 2/3$ , we have  $q = \frac{1}{H} < \frac{1}{2-2H}$ . In this case, we see that the (5.8) implies (5.7) when we choose  $p$  close to  $\frac{1}{2-2H}$ . This means that the condition (5.7) is satisfied for many random variables we encounter when  $H > 2/3$ .

Before we proceed to solve (5.5), we state some assumptions on the coefficients  $b$  and  $\sigma$ .

**Hypothesis 5.2.** Let  $\mathcal{L}$  be a positive constant. The measurable functions  $b, \sigma : \mathcal{T} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  satisfy the following conditions.

(i)  $b$  is continuously differentiable in  $x$  and satisfies

$$\begin{cases} |b(t, x, \omega)| \leq \mathcal{L}(1 + |x|); \\ \left| \frac{\partial}{\partial x} b(t, x, \omega) \right| \leq \mathcal{L}; \\ \|\mathbb{D}b(t, x, \omega)\|_{\mathcal{H}} \leq \mathcal{L}. \end{cases}$$

- (ii)  $\sigma(t, x, \omega)$  is twice continuously differentiable in  $x$  with bounded first and second derivative and satisfies

$$\begin{cases} |\sigma(t, x, \omega)| \leq \mathcal{L}(1 + |x|); \\ |\frac{\partial}{\partial t}\sigma(t, x, \omega) - \frac{\partial}{\partial t}\sigma(t, y, \tilde{\omega})| \leq \mathcal{L}(|x - y| + \|\omega - \tilde{\omega}\|_{\mathcal{H}}); \\ |\frac{\partial}{\partial x}\sigma(t, x, \omega)| + |\frac{\partial^2}{\partial x^2}\sigma(t, x, \omega)| \leq \mathcal{L}; \\ |\mathbb{D}_t^\phi \sigma(t, x, \omega)| \leq \mathcal{L}(1 + |x|); \\ \|\mathbb{D}\sigma(t, x, \omega)\|_{\mathcal{H}} + \|\mathbb{D}^2\sigma(t, x, \omega)\|_{\mathcal{H}^2} \leq \mathcal{L}; \\ \|\mathbb{D}\frac{\partial}{\partial x}\sigma(t, x, \omega)\|_{\mathcal{H}} \leq \mathcal{L}. \end{cases}$$

From now on we denote  $\mathbb{B} = \mathcal{H} \oplus \mathbb{R}$  and we denote by  $\mathbb{B}[0, T]$  the space of all continuous functions from  $[0, T]$  to  $\mathbb{B}$ . Similarly, we will also use the notation  $\mathcal{H}[0, T]$ .  $\mathbb{R}[0, T]$  is then the space  $C([0, T])$  of all continuous functions from  $[0, T]$  to  $\mathbb{R}$ .

Define a mapping from  $\mathbb{B}[0, T]$  to  $\mathbb{B}[0, T]$  as follows.

$$\begin{cases} F_1(t, \Gamma, z) := \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s) + \omega) \int_0^s \phi(s, u) du ds; \\ F_2(t, \Gamma, z) := \eta(\omega) + \int_0^t \tilde{b}(s, z(s), \Gamma(s) + \omega) ds + \int_0^t \sigma(s, z(s), \Gamma(s) + \omega) \delta B(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z(s), \Gamma(s) + \omega) \tilde{\sigma}(u, z(u), \Gamma(u) + \omega) \phi(s, u) du ds. \end{cases} \quad (5.9)$$

We also write  $F_i(\Gamma, z) = F_i(t, \Gamma, z)$ ,  $i = 1, 2$ . It is easy to see that for any  $(\Gamma, z) \in \mathbb{B}[0, T]$ ,  $(F_1(\Gamma, z), F_2(\Gamma, z))$  is also in  $\mathbb{B}[0, T]$ .

**Lemma 5.3.** *For any  $\tau \in \mathcal{T}$ , if  $(\Gamma, z) \in \mathbb{B}[0, \tau]$ , then  $F_1(\Gamma, z) \in \mathcal{H}[0, \tau]$  and*

$$\left\| \frac{d}{dt} F_1(\Gamma, z) \right\|_{\mathcal{H}} \leq \kappa, \quad (5.10)$$

$$\|F_1(\Gamma, z)\|_{0, \tau} \leq \kappa \tau, \quad (5.11)$$

where and in what follows  $\kappa = c_{p, H, T, \mathcal{L}}$  is a constant depending only on  $p, H, T$  and the bound  $\mathcal{L}$  of the coefficients  $b$  and  $\sigma$ , which may vary at different occurrences.

*Proof* From the definition of  $F_1$  we see

$$\frac{d}{dt} F_1(\Gamma, z) = \tilde{\sigma}(t, z(t), \Gamma(t) + \omega) \int_0^t \phi(t, u) du.$$

Recall that we fix  $p < \frac{1}{2-2H}$ . Since  $\tilde{\sigma}$  is bounded and  $\phi(t, u) = H(2H-1)|t-u|^{2H-2}$ , we have

$$\left\| \frac{d}{dt} F_1(\Gamma, z) \right\|_{\mathcal{H}}^p \leq \kappa \int_0^T \phi(t, u)^p du = \kappa.$$

This proves (5.10). Similarly, since  $F_1(0) = 0$ , we have

$$\|F_1(\Gamma, z)\|_{0, \tau} \leq \left\| \frac{d}{dt} F_1(\Gamma, z) \right\|_{0, \tau} \tau = \kappa \tau,$$

proving (5.11). ■

**Lemma 5.4.** *For any  $\tau \in \mathcal{T}$ , if  $(\Gamma, z) \in \mathbb{B}[0, \tau]$ , then  $F_2(\Gamma, z) \in C[0, \tau]$  and for any  $0 \leq a < b \leq \tau$ ,*

$$\begin{aligned} \|F_2\|_{a, b, \beta} &\leq \kappa (1 + \|B\|_{a, b, \beta}) \left\{ 1 + \|z\|_{0, a} \right. \\ &\quad \left. + \|z\|_{a, b, \beta} (b-a)^\beta + \|\Gamma\|_{a, b, \beta} (b-a)^\beta \right\}. \end{aligned} \quad (5.12)$$

*Proof* First, we write

$$F_2 = \eta(\omega) + F_{21} + F_{22} + F_{23},$$

where

$$\begin{cases} F_{21}(t) = \int_0^t \tilde{b}(s, z(s), \Gamma(s) + \omega) ds; \\ F_{22}(t) = \int_0^t \sigma(s, z(s), \Gamma(s) + \omega) \delta B(s); \\ F_{23}(t) = \int_0^t \int_0^s \sigma(s, z(s), \Gamma(s) + \omega) \tilde{\sigma}(u, z(u), \Gamma(u) + \omega) \phi(s, u) duds. \end{cases}$$

From the assumption on  $b$  and  $\mathbb{D}_s^\phi \sigma$ , we see that for any  $0 \leq a < b \leq \tau$ , we have

$$\begin{aligned} |F_{21}(b) - F_{21}(a)| &= \int_a^b |\tilde{b}(s, z(s), \Gamma(s) + \omega)| ds \\ &\leq \kappa \int_a^b [1 + |z(s)|] ds \\ &\leq \kappa [1 + \|z\|_{a,b}] (b - a). \end{aligned}$$

This implies that

$$\begin{aligned} \|F_{21}\|_{a,b,\beta} &\leq \kappa [1 + \|z\|_{a,b}] \\ &\leq \kappa [1 + |z(a)| + \|z\|_{a,b,\beta} (b - a)^\beta]. \end{aligned} \quad (5.13)$$

Now we consider  $F_{23}$ . We have

$$\begin{aligned} |F_{23}(b) - F_{23}(a)| &= \int_a^b \int_0^s |\sigma(s, z(s), \Gamma(s) + \omega) \tilde{\sigma}(u, z(u), \Gamma(u) + \omega) \phi(s, u)| duds \\ &\leq \kappa \int_a^b \int_0^s [1 + |z(s)|] \phi(s, u) duds \\ &\leq \kappa [1 + \|z\|_{a,b}] \int_a^b \int_0^s \phi(s, u) duds \\ &\leq \kappa [1 + \|z\|_{a,b}] (b^{2H} - a^{2H}) \\ &\leq \kappa [1 + \|z\|_{a,b}] (b - a). \end{aligned}$$

This implies

$$\|F_{23}\|_{a,b,\beta} \leq \kappa [1 + \|z\|_{0,a} + \|z\|_{a,b} (b - a)^\beta]. \quad (5.14)$$

$F_{22}$  is more complicated to handle because the fractional Brownian motion  $B$  is not differentiable. Denote  $\sigma_r = \sigma(r, z(r), \Gamma(r) + \omega)$ . We have for an  $\alpha \in (1 - \beta, \beta)$ ,

$$\begin{aligned} \left| \int_a^b \sigma(r, z(r), \Gamma(r) + \omega) \delta B(r) \right| &= \left| \int_a^b D_{b-}^{1-\alpha} B_{b-}(r) D_{a+}^\alpha \sigma_r dr \right| \\ &\leq \kappa \|B\|_{a,b,\beta} \left| \int_a^b (b - r)^{\alpha+\beta-1} \left\{ \frac{\sigma_r}{(r - a)^\alpha} + \int_a^r \frac{\sigma_r - \sigma_\rho}{(r - \rho)^{\alpha+1}} d\rho \right\} dr \right| \\ &\leq \kappa \|B\|_{a,b,\beta} \left\{ (1 + \|z\|_{a,b}) (b - a)^\beta + (\|z\|_{a,b,\beta} + \|\Gamma\|_{a,b,\beta}) (b - a)^{2\beta} \right\}. \end{aligned}$$

This implies

$$\|F_{22}\|_{a,b,\beta} \leq \kappa \|B\|_{a,b,\beta} \left\{ 1 + \|z\|_{0,a} + (\|z\|_{a,b,\beta} + \|\Gamma\|_{a,b,\beta}) (b - a)^\beta \right\}. \quad (5.15)$$

Combining (5.13), (5.14) and (5.15), we prove the lemma. ■

To bound the Hölder norm of the difference, we first need the following simple general result.

**Lemma 5.5.** *Let  $B_1$  and  $B_2$  be two Banach spaces with norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  and let  $f : B_1 \rightarrow B_2$  be twice continuously (Frechet) differentiable with bounded first and second derivatives.*

(i) *If  $x_1, x_2, y_1, y_2 \in B_1$ , then*

$$\begin{aligned} \|f(y_2) - f(y_1) - f(x_2) + f(x_1)\|_2 &\leq \|f'\|_\infty \|y_2 - y_1 - x_2 + x_1\|_1 \\ &\quad + \|f''\|_\infty [\|y_1 - x_1\|_1 + \|y_2 - x_2\|_1] \|x_2 - x_1\|_1. \end{aligned} \quad (5.16)$$

(ii) *Let  $x_1, x_2 : [a, b] \rightarrow B_1$  be Hölder continuous of order  $\beta$ . Then for any  $a \leq s < t \leq b$ , we have*

$$\begin{aligned} |f(x_2(t)) - f(x_1(t)) - f(x_2(s)) + f(x_1(s))|_2 &\leq \|f'\|_\infty \|x_2 - x_1\|_{s,r,\beta} (r-s)^\beta \\ &\quad + \|f''\|_\infty [\|x_1\|_{s,r,\beta} + \|x_2\|_{s,r,\beta}] \|x_2 - x_1\|_{s,r} (r-s)^\beta. \end{aligned} \quad (5.17)$$

*Proof* This inequality may be well-known. We include a short proof for the completeness. Using the mean value theorem we have

$$\begin{aligned} &f(y_2) - f(y_1) - f(x_2) + f(x_1) \\ &= \int_0^1 f'((1-\theta)y_1 + \theta y_2) d\theta (y_2 - y_1) \\ &\quad - \int_0^1 f'((1-\theta)x_1 + \theta x_2) d\theta (x_2 - x_1) \\ &= \int_0^1 f'((1-\theta)y_1 + \theta y_2) d\theta (y_2 - y_1 - x_2 + x_1) \\ &\quad + \int_0^1 [f'((1-\theta)y_1 + \theta y_2) - f'((1-\theta)x_1 + \theta x_2)] d\theta (x_2 - x_1) \\ &= \int_0^1 f'(y_1 + \theta(y_2 - y_1)) d\theta (y_2 - y_1 - x_2 + x_1) \\ &\quad + \int_0^1 \int_0^1 f''(v[y_1 + \theta(y_2 - y_1)] + (1-v)[(1-\theta)x_1 + \theta x_2]) d\theta dv \\ &\quad (x_2 - x_1) \otimes [(1-\theta)(y_1 - x_1) + \theta(y_2 - x_2)]. \end{aligned}$$

This proves (5.16) easily. The inequality (5.17) is straightforward consequences of (5.16). ■

**Lemma 5.6.** *Denote*

$$F_j^{(i)}(t) = F_j(t, \Gamma_i, z_i), \quad i, j = 1, 2.$$

*Then, we have*

$$\left\| \frac{d}{dt} F_1^{(2)}(t) - \frac{d}{dt} F_1^{(1)}(t) \right\|_{\mathcal{H}} \leq \kappa [\|z_2 - z_1\|_{0,\tau} + \|\Gamma_2 - \Gamma_1\|_{0,\tau}] \quad (5.18)$$

and

$$\begin{aligned}
\left\| F_2^{(2)} - F_2^{(1)} \right\|_{a,b,\beta} &\leq \kappa(1 + \|B\|_{a,b,\beta})(1 + \|z_1\|_{0,a} + \|z_2\|_{0,a} + \|z_1\|_{a,b,\beta} \\
&\quad + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}) \left[ |z_2(a) - z_1(a)| + \|\Gamma_2(a) - \Gamma_1(a)\| \right. \\
&\quad \left. + [\|z_2 - z_1\|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta}] (b-a)^\beta \right]. \tag{5.19}
\end{aligned}$$

*Proof* Let  $(\Gamma_1, z_1)$  and  $(\Gamma_2, z_2)$  be two elements in  $\mathbb{B}[0, \tau]$ . We recall

$$\begin{cases} F_1^{(i)}(t) = \omega + \int_0^t \tilde{\sigma}(s, z_i(s), \Gamma_i(s) + \omega) \int_0^s \phi(s, u) du ds; \\ F_2^{(i)}(t) = \eta(\omega) + \int_0^t \tilde{b}(s, z_i(s), \Gamma_i(s) + \omega) ds + \int_0^t \sigma(s, z_i(s), \Gamma_i(s) + \omega) \delta B(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z_i(s), \Gamma_i(s) + \omega) \tilde{\sigma}(u, z_i(u), \Gamma_i(u) + \omega) \phi(s, u) du ds. \end{cases}$$

To simplify notation we also denote

$$\begin{aligned} b^{(i)}(s) &= b(s, z_i(s), \Gamma_i(s) + \omega), & \tilde{b}^{(i)}(s) &= \tilde{b}(s, z_i(s), \Gamma_i(s) + \omega), \\ \sigma^{(i)}(s) &= \sigma(s, z_i(s), \Gamma_i(s) + \omega), & \tilde{\sigma}^{(i)}(s) &= \tilde{\sigma}(s, z_i(s), \Gamma_i(s) + \omega). \end{aligned}$$

We have for any  $t \in [0, \tau]$ ,

$$\begin{aligned} \left\| \frac{d}{dt} F_1^{(2)}(t) - \frac{d}{dt} F_1^{(1)}(t) \right\|_{\mathcal{H}} &= \left\| \int_0^t [\tilde{\sigma}^{(2)}(t) - \tilde{\sigma}^{(1)}(t)] \phi(t, u) du \right\|_{\mathcal{H}} \\ &\leq \kappa \left\| \int_0^t [|z_2(t) - z_1(t)| + \|\Gamma_2(t) - \Gamma_1(t)\|_{\mathcal{H}}] \phi(t, u) du \right\|_{\mathcal{H}} \\ &\leq \kappa [\|z_2 - z_1\|_{0,\tau} + \|\Gamma_2(t) - \Gamma_1(t)\|_{0,\tau}] \left[ \int_0^T \phi(t, u)^p du \right]^{1/p} \\ &\leq \kappa [\|z_2 - z_1\|_{0,\tau} + \|\Gamma_2(t) - \Gamma_1(t)\|_{0,\tau}]. \end{aligned}$$

This is (5.18).

As in the proof of Lemma 5.3 we denote

$$\begin{cases} F_{21}^{(i)}(t) = \int_0^t \tilde{b}(s, z_i(s), \Gamma_i(s) + \omega) ds \\ F_{22}^{(i)}(t) = \int_0^t \sigma(s, z_i(s), \Gamma_i(s) + \omega) \delta B(s) \\ F_{23}^{(i)}(t) = \int_0^t \int_0^s \sigma(s, z_i(s), \Gamma_i(s) + \omega) \tilde{\sigma}(u, z_i(u), \Gamma_i(u) + \omega) \phi(s, u) du ds. \end{cases}$$

For any  $a, b \in [0, \tau]$ , we have

$$\begin{aligned} &|F_{21}^{(2)}(b) - F_{21}^{(1)}(b) - F_{21}^{(2)}(a) + F_{21}^{(1)}(a)| \\ &= \int_a^b |\tilde{b}_2(r) - \tilde{b}_1(r)| dr \\ &\leq \kappa \int_a^b [|z_2(r) - z_1(r)| + \|\Gamma_2(r) - \Gamma_1(r)\|_{\mathcal{H}}] dr \\ &\leq \kappa [|z_2 - z_1|_{a,b} + \|\Gamma_2 - \Gamma_1\|_{a,b}] (b-a). \end{aligned}$$

This yields

$$\begin{aligned}
\|F_{21}^{(2)} - F_{21}^{(1)}\|_{a,b,\beta} &\leq \kappa [|z_2 - z_1|_{a,b} + \|\Gamma_2 - \Gamma_1\|_{a,b}] (b-a)^{1-\beta} \\
&\leq \kappa [|z_2(a) - z_1(a)| + \|\Gamma_2(a) - \Gamma_1(a)\|] (b-a)^{1-\beta} \\
&\quad + \kappa [|z_2 - z_1|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta}] (b-a) \\
&\leq \kappa [|z_2(a) - z_1(a)| + \|\Gamma_2(a) - \Gamma_1(a)\|] \\
&\quad + \kappa [|z_2 - z_1|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta}] (b-a)^\beta. \quad (5.20)
\end{aligned}$$

Now we find the bounds for  $F_{22}^{(i)}$ . We have

$$\begin{aligned}
&|F_{22}^{(2)}(b) - F_{22}^{(1)}(b) - F_{22}^{(2)}(a) + F_{22}^{(1)}(a)| \\
&= \left| \int_a^b (\sigma^{(2)}(r) - \sigma^{(1)}(r)) \delta B(r) \right| \\
&= \left| \int_a^b D_{b-}^{1-\alpha} B_{t-}(r) D_{a+}^\alpha (\sigma^{(2)}(r) - \sigma^{(1)}(r)) dr \right| \\
&= \frac{1}{\Gamma(1-\alpha)} \left| \int_a^b D_{b-}^{1-\alpha} B_{t-}(r) \left( \frac{\sigma^{(2)}(r) - \sigma^{(1)}(r)}{(r-a)^\alpha} \right. \right. \\
&\quad \left. \left. + \alpha \int_a^r \frac{\sigma^{(2)}(r) - \sigma^{(1)}(r) - \sigma^{(2)}(\rho) + \sigma^{(1)}(\rho)}{(r-\rho)^{\alpha+1}} d\rho \right) dr \right| \\
&\leq \kappa \|B\|_{a,b,\beta} [I_1 + I_2], \quad (5.21)
\end{aligned}$$

where

$$I_1 = \int_a^b (b-r)^{\alpha+\beta-1} \frac{|\sigma^{(2)}(r) - \sigma^{(1)}(r)|}{(r-a)^\alpha} dr; \quad (5.22)$$

$$I_2 = \int_a^b (b-r)^{\alpha+\beta-1} \int_a^r \frac{|\sigma^{(2)}(r) - \sigma^{(1)}(r) - \sigma^{(2)}(\rho) + \sigma^{(1)}(\rho)|}{(r-\rho)^{\alpha+1}} d\rho dr. \quad (5.23)$$

It is easy to see that

$$I_1 \leq \kappa [|z_2 - z_1|_{a,b} + \|\Gamma_2 - \Gamma_1\|_{a,b}] (b-a)^\beta. \quad (5.24)$$

To bound  $I_2$ , we need the following identity.

$$\sigma^{(2)}(r) - \sigma^{(1)}(r) - \sigma^{(2)}(\rho) + \sigma^{(1)}(\rho) = J_1 + J_2 + J_3 + J_4 + J_5, \quad (5.25)$$



where

$$\begin{cases} J_1 = \sigma(r, z_2(r), \Gamma_2(r) + \omega) - \sigma(r, z_1(r), \Gamma_2(r) + \omega) \\ \quad - \sigma(r, z_2(\rho), \Gamma_2(r) + \omega) + \sigma(r, z_1(\rho), \Gamma_2(r) + \omega); \\ J_2 = \sigma(r, z_1(r), \Gamma_2(r) + \omega) - \sigma(r, z_1(r), \Gamma_1(r) + \omega) \\ \quad - \sigma(r, z_1(r), \Gamma_2(\rho) + \omega) + \sigma(r, z_1(r), \Gamma_1(\rho) + \omega); \\ J_3 = \sigma(r, z_1(r), \Gamma_2(\rho) + \omega) - \sigma(r, z_1(r), \Gamma_1(\rho) + \omega) \\ \quad - \sigma(r, z_1(\rho), \Gamma_2(\rho) + \omega) + \sigma(r, z_1(\rho), \Gamma_1(\rho) + \omega); \\ J_4 = \sigma(r, z_1(\rho), \Gamma_2(\rho) + \omega) - \sigma(r, z_1(\rho), \Gamma_2(r) + \omega) \\ \quad - \sigma(r, z_2(\rho), \Gamma_2(\rho) + \omega) + \sigma(r, z_2(\rho), \Gamma_2(r) + \omega); \\ J_5 = \sigma(r, z_2(\rho), \Gamma_2(\rho) + \omega) - \sigma(\rho, z_2(\rho), \Gamma_2(\rho) + \omega) \\ \quad + \sigma(\rho, z_1(\rho), \Gamma_1(\rho) + \omega) - \sigma(r, z_1(\rho), \Gamma_1(\rho) + \omega). \end{cases}$$

From Lemma 5.5, we see that

$$|J_1| \leq \kappa [\|z_2 - z_1\|_{\rho, r, \beta} + (\|z_1\|_{\rho, r, \beta} + \|z_2\|_{\rho, r, \beta}) \|z_2 - z_1\|_{\rho, r}] (r - \rho)^\beta \quad (5.26)$$

and

$$\|J_2\| \leq \kappa [\|\Gamma_2 - \Gamma_1\|_{\rho, r, \beta} + (\|\Gamma_1\|_{\rho, r, \beta} + \|\Gamma_2\|_{\rho, r, \beta}) \|\Gamma_2 - \Gamma_1\|_{\rho, r, \beta}] (r - \rho)^\beta. \quad (5.27)$$

Use the mean value theorem to obtain

$$\begin{aligned} J_3 &= \int_0^1 \int_0^1 \mathbb{D}\tilde{\sigma}((1-v)z_1(\rho) + vz_1(r), (1-\theta)\Gamma_1(\rho) + \theta\Gamma_2(\rho)) d\theta dv \\ &\quad (z_1(r) - z_1(\rho))(\Gamma_2(\rho) - \Gamma_1(\rho)). \end{aligned}$$

This shows

$$|J_3| \leq \kappa \|z_1\|_{\rho, r, \beta} \|\Gamma_2 - \Gamma_1\|_{\rho, r} (r - \rho)^\beta. \quad (5.28)$$

In a similar way we can obtain

$$|J_4| \leq \kappa \|\dot{\Gamma}_2\|_{\rho, r} \|z_2 - z_1\|_{\rho, r} (r - \rho). \quad (5.29)$$

From the assumption on  $\sigma$  it is to verify that

$$|J_5| \leq \kappa (r - \rho) [\|z_2 - z_1\|_{\rho, r} + \|\Gamma_2 - \Gamma_1\|_{\rho, r}]. \quad (5.30)$$

Combining (5.26)-(5.30), we have

$$\begin{aligned} & \left| \sigma^{(2)}(r) - \sigma^{(1)}(r) - \sigma^{(2)}(\rho) + \sigma^{(1)}(\rho) \right| \\ & \leq \kappa \left[ \|z_2 - z_1\|_{a, b, \beta} + \|\Gamma_2 - \Gamma_1\|_{a, b, \beta} \right. \\ & \quad + (1 + \|z_1\|_{a, b, \beta} + \|z_2\|_{a, b, \beta} + \|\Gamma_1\|_{a, b, \beta} + \|\Gamma_2\|_{a, b, \beta}) \|z_2 - z_1\|_{a, b} \\ & \quad + (1 + \|z_1\|_{a, b, \beta} + \|z_2\|_{a, b, \beta} + \|\Gamma_1\|_{a, b, \beta} \\ & \quad \left. + \|\Gamma_2\|_{a, b, \beta}) \|\Gamma_2 - \Gamma_1\|_{a, b} \right] (r - \rho)^\beta. \end{aligned} \quad (5.31)$$

Substituting the above inequality into (5.23) yields

$$\begin{aligned}
I_2 \leq & \kappa \left[ \|z_2 - z_1\|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta} \right. \\
& + (1 + \|z_1\|_{a,b,\beta} + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}) \|z_2 - z_1\|_{a,b} \\
& \left. + (1 + \|z_1\|_{a,b,\beta} + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}) \|\Gamma_2 - \Gamma_1\|_{a,b} \right] (b-a)^{2\beta}.
\end{aligned} \tag{5.32}$$

Substituting the bounds for  $I_1$  and  $I_2$  into (5.21) we have

$$\begin{aligned}
\|F_{22}^{(2)} - F_{22}^{(1)}\|_{a,b,\beta} & \leq \kappa \|B\|_{a,b,\beta} \left[ \|z_2 - z_1\|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta} \right. \\
& + (1 + \|z_1\|_{a,b,\beta} + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}) \|z_2 - z_1\|_{a,b} \\
& \left. + (1 + \|z_1\|_{a,b,\beta} + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}) \|\Gamma_2 - \Gamma_1\|_{a,b} \right] (b-a)^\beta \\
& \leq \kappa \|B\|_{a,b,\beta} (1 + \|z_1\|_{a,b,\beta} + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}) \\
& \quad \left[ |z_2(a) - z_1(a)| + \|\Gamma_2(a) - \Gamma_1(a)\| \right. \\
& \quad \left. + [\|z_2 - z_1\|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta}] (b-a)^\beta \right].
\end{aligned} \tag{5.33}$$

Finally, we turn to bound  $\|F_{23}^2 - F_{23}^1\|_{a,b,\beta}$ . We have

$$\begin{aligned}
& |F_{23}^2(b) - F_{23}^1(b) - F_{23}^2(a) + F_{23}^1(a)| \\
& \leq \int_a^b \int_0^s |\sigma(s, z_2(s), \Gamma_2(s)) - \sigma(s, z_1(s), \Gamma_1(s))| |\tilde{\sigma}(u, z_2(u), \Gamma_2(u))| dud s \\
& \quad + \int_a^b \int_0^s |\tilde{\sigma}(u, z_2(u), \Gamma_2(u)) - \tilde{\sigma}(u, z_1(u), \Gamma_1(u))| |\sigma(s, z_1(s), \Gamma_1(s))| dud s \\
& \leq \kappa(b-a) \{ \|z_2 - z_1\|_{a,b} + \|\Gamma_2 - \Gamma_1\|_{a,b} + \|z_1\|_{a,b} [\|z_2 - z_1\|_{0,b} + \|\Gamma_2 - \Gamma_1\|_{0,b}] \}.
\end{aligned}$$

This means

$$\begin{aligned}
\|F_{23}^2 - F_{23}^1\|_{a,b,\beta} & \leq \kappa(b-a)^{1-\beta} \{ \|z_2 - z_1\|_{a,b} + \|\Gamma_2 - \Gamma_1\|_{a,b} \\
& + \|z_1\|_{a,b} [\|z_2 - z_1\|_{0,b} + \|\Gamma_2 - \Gamma_1\|_{0,b}] \} \\
& \leq \kappa(1 + \|z_1\|_{a,b}) [\|z_2 - z_1\|_{0,b} + \|\Gamma_2 - \Gamma_1\|_{0,b}] (b-a)^{1-\beta} \\
& \leq \kappa(b-a)^{1-\beta} (1 + \|z_1\|_{a,b}) \left[ \|z_2 - z_1\|_{0,a} + \|\Gamma_2 - \Gamma_1\|_{0,a} \right. \\
& \quad \left. + [\|z_2 - z_1\|_{a,b,\beta} + \|\Gamma_2 - \Gamma_1\|_{a,b,\beta}] (b-a)^\beta \right].
\end{aligned} \tag{5.34}$$

Combining (5.20), (5.33), and (5.34), we prove (5.19). ■

Now we are ready to prove one of our main theorems of this section.

**Theorem 5.7.** *Let  $T \in (0, \infty)$  be any given number. Assume the hypothesis 5.2. Then, the equation (5.4) has a unique solution. Moreover, there is a  $\tau_0 > 0$  such*

that the solution satisfies

$$\sup_{0 \leq t \leq T} |z(t)| \leq c_2 \exp \left\{ c_1 \|B\|_{0,T,\beta}^{1/\beta} \right\} \quad (5.35)$$

$$\sup_{0 \leq a < b \leq T, b-a \leq \tau_0} |z|_{a,b,\beta} \leq c_2 \exp \left\{ c_1 \|B\|_{0,T,\beta}^{1/\beta} \right\} \quad (5.36)$$

for some constants  $c_1$  and  $c_2$  dependent only on  $\beta, p, T$  and the bound  $\mathcal{L}$  for the coefficients  $b$  and  $\sigma$ .

*Proof* Let  $\mathbb{B} = \mathcal{H} \oplus \mathbb{R}$ . Then  $F = (F_1, F_2)$  defined by (5.9) is a mapping from (some domain of)  $\mathbb{B}[0, T]$  to  $\mathbb{B}[0, T]$ . The inequality (5.10) implies that there is a constant  $c$  depending only on  $\beta, p, T$  and the bound  $\mathcal{L}$  for the coefficients  $b$  and  $\sigma$  such that

$$\|F_1\|_{a,b,\beta} \leq c$$

for any  $a, b \in [0, T]$ . This together with (5.12) implies that  $F = (F_1, F_2)$  satisfies the condition (i) of Theorem 4.1 with  $\kappa$  there being replaced by  $c(1 + \|B\|_{a,b,\beta})$ . Lemma 5.6 implies that  $F = (F_1, F_2)$  satisfies the condition (ii) of Theorem 4.1 with the function  $\bar{h}$  being given by

$$\begin{aligned} \bar{h} &= \bar{h}((\Gamma_1, z_1), (\Gamma_2, z_2)) = \mathcal{L}(1 + \|B\|_{a,b,\beta})(1 + \|z_1\|_{0,a} + \|z_2\|_{0,a} \\ &\quad + \|z_1\|_{a,b,\beta} + \|z_2\|_{a,b,\beta} + \|\Gamma_1\|_{a,b,\beta} + \|\Gamma_2\|_{a,b,\beta}). \end{aligned}$$

Thus, we can apply Theorem 4.1 to prove that there is a  $x \in \mathbb{B}[0, T]$  satisfies the equation  $x(t) = F(t, x)$ . This means that  $x$  satisfies the equation (5.5), hence it satisfies (5.4). The bounds (5.35) and (5.36) are immediate consequence of (4.4).  $\blacksquare$

**5.3. Solution to the original equation.** We need the following lemmas in the proof of the existence and uniqueness theorem for equation (5.4).

**Lemma 5.8.** *Let  $\Gamma : \mathcal{T} \times \Omega \rightarrow \Omega$  be continuously differentiable in  $t$  and  $\mathcal{H}$ -differentiable in  $\omega$ . If  $\Gamma(t) : \Omega \rightarrow \Omega$  has an inverse  $\Lambda(t)$  and if  $\Lambda(t)$  is differentiable in  $t$  in the Hilbert space  $\mathcal{H}$ , then*

$$\frac{\partial \Lambda}{\partial t}(t, \omega) = -(\mathbb{D}\Lambda)(t, \omega) \frac{\partial \Gamma}{\partial t}(t, \Lambda(t, \omega)). \quad (5.37)$$

*Proof* Since  $\Lambda(t)$  is the inverse of  $\Gamma(t)$  we have

$$\Gamma(t, \Lambda(t, \omega)) = \omega, \quad \forall \omega \in \Omega.$$

Differentiating both sides with respect to  $\omega$ , we have

$$(\mathbb{D}\Gamma)(t, \Lambda(t, \omega))(\mathbb{D}\Lambda)(t, \omega) = I,$$

where  $I$  is an identity operator from  $\mathcal{H}$  to  $\mathcal{H}$ . Therefore, we obtain

$$[(\mathbb{D}\Gamma)(t, \Lambda(t, \omega))]^{-1} = (\mathbb{D}\Lambda)(t, \omega). \quad (5.38)$$

On the other hand, differentiating  $\Gamma(t, \Lambda(t, \omega)) = \omega$  with respect to  $t$ , we have

$$\frac{\partial \Gamma}{\partial t}(t, \omega) + (\mathbb{D}\Gamma)(t, \Lambda(t, \omega)) \frac{\partial \Lambda}{\partial t}(t, \omega) = 0.$$

Thus

$$\frac{\partial \Lambda}{\partial t}(t, \omega) = -[(\mathbb{D}\Gamma)(t, \Lambda(t, \omega))]^{-1} \frac{\partial \Gamma}{\partial t}(t, \Lambda(t, \omega)). \quad (5.39)$$

Combining (5.38) and (5.39) we have

$$\frac{\partial \Lambda}{\partial t}(t, \omega) = -(\mathbb{D}\Lambda)(t, \omega) \frac{\partial \Gamma}{\partial t}(t, \Lambda(t, \omega)). \quad (5.40)$$

This proves the lemma. ■

**Lemma 5.9.** *Let  $\tau \in (0, T]$  be a positive number. Assume that  $\Gamma(t) : \Omega \rightarrow \Omega$  defined by (5.4) has an inverse  $\Lambda(t)$  for all  $t \in [0, \tau]$  and assume that  $\Lambda(t)$  is differentiable in  $t \in [0, \tau]$  in the Hilbert space  $\mathcal{H}$ . Let  $z$  be defined by (5.4). Then  $x(t) = z(t, \Lambda(t))$ ,  $t \in [0, \tau]$  satisfies the equation (5.2).*

*Proof* If  $\Gamma(t)$  defined by (5.4) has inverse  $\Lambda(t)$ , then

$$\omega = \Gamma(t, \Lambda(t)) = \Lambda(t) + \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s)) \int_0^\cdot \phi(s, u) du ds \Big|_{\omega=\Lambda(t)}.$$

Or

$$\begin{cases} \Lambda(t) = \omega + \int_0^\cdot h(t, u, \omega) du & \text{with} \\ h(t, u, \omega) = - \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s)) \phi(s, u) ds \Big|_{\omega=\Lambda(t)}. \end{cases} \quad (5.41)$$

On the other hand, from (5.4) we have

$$\begin{aligned} \frac{d}{dt} \Gamma(t, \omega) \Big|_{\omega=\Lambda(t)} &= \tilde{\sigma}(t, z(t), \Gamma(t)) \int_0^\cdot \phi(t, u) du \Big|_{\omega=\Lambda(t)} \\ &= \tilde{\sigma}(t, x(t, \omega), \omega) \int_0^\cdot \phi(t, u) du. \end{aligned} \quad (5.42)$$

We apply the Itô formula (3.5) to  $z(t, \Lambda(t))$  with

$$\begin{aligned} f_0 &= \tilde{b}(s, z(s), \Gamma(s)) + \sigma(s, z(s), \Gamma(s)) \int_0^s \tilde{\sigma}(u, z(u), \Gamma(u)) \phi(s, u) du \\ f_1 &= \sigma(s, z(s), \Gamma(s)) \end{aligned}$$

and with  $h$  defined by (5.41). We shall use  $\sigma(s, x(s))$  to denote  $\sigma(s, x(s, \omega), \omega)$  etc. Noticing  $z(s) \Big|_{\omega=\Lambda(s)} = x(s)$ , we have

$$\begin{aligned} x(t) &= z(t, \Lambda(t)) \\ &= \eta(\omega) + \int_0^t \left( \int_0^s \tilde{\sigma}(u, z(u), \Gamma(u)) \phi(s, u) du \right) \Big|_{\omega=\Lambda(s)} \sigma(s, x(s)) ds \\ &\quad + \int_0^t \tilde{b}(s, x(s)) ds + \int_0^t \sigma(s, x(s)) \delta B(s) \\ &\quad + \int_0^t \mathbb{D}z(s, \Lambda(s)) \frac{d}{ds} \Lambda(s) ds + \int_0^t \sigma(s, x(s)) h(s, s, \omega) ds \\ &= \eta(\omega) + I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (5.43)$$

From (5.41) and since  $\phi(s, u) = \phi(u, s)$ , we see that

$$I_1 + I_5 = 0. \quad (5.44)$$

By Lemma 5.8 and then by (5.42), we have

$$\begin{aligned}
I_4 &= \int_0^t (\mathbb{D}z)(s, \Lambda(s)) \frac{\partial}{\partial s} \Lambda(s) ds \\
&= - \int_0^t (\mathbb{D}z)(s, \Lambda(s)) (\mathbb{D}\Lambda)(s) \left( \frac{\partial}{\partial s} \Gamma \right) (s, \Lambda(s)) ds \\
&= - \int_0^t [\mathbb{D}x(s)] \tilde{\sigma}(s, x(s)) \int_0^\cdot \phi(s, u) du ds.
\end{aligned}$$

This yields

$$I_4 = - \int_0^t \tilde{\sigma}(s, x(s)) \mathbb{D}_s^\phi x(s) ds. \quad (5.45)$$

Substituting (5.44) and (5.45) into (5.43) we have

$$\begin{aligned}
x(t) &= \eta(\omega) + \int_0^t \tilde{b}(s, x(s)) ds + \int_0^t \sigma(s, x(s)) \delta B(s) \\
&\quad - \int_0^t \tilde{\sigma}(s, x(s)) \mathbb{D}_s^\phi x(s) ds.
\end{aligned} \quad (5.46)$$

This is the lemma. ■

Now we show that there is a positive  $\tau$  such that  $\Gamma(t)$  has inverse  $\Lambda(t)$ . Before we continue, we need the following simple inequality.

**Lemma 5.10.** *Assume that  $B : [0, T] \rightarrow \mathbb{R}$  is a Hölder continuous function of exponent  $\beta \in (0, 1)$ . Let  $\mathbb{B}_1$  and  $\mathbb{B}_2$  be two Banach spaces and let  $f : [0, T] \rightarrow \mathbb{B}_1$  and  $g : [0, T] \rightarrow \mathbb{B}_2$  be two Hölder continuous functions with exponent  $\alpha \in (1 - \beta, 1)$ . Then*

$$\begin{aligned}
\left\| \int_0^\cdot f(s) \otimes g(s) dB(s) \right\|_{a,b,\beta} &\leq \kappa \|B\|_{a,b,\beta} \left\{ \|f\|_{a,b} \|g\|_{a,b} \right. \\
&\quad \left. + [\|f\|_{a,b} \|g\|_{a,b,\beta} + \|g\|_{a,b} \|f\|_{a,b,\beta}] (b-a)^\beta \right\}. \quad (5.47)
\end{aligned}$$

*Proof* We refer to [10], and in particular, the references therein for the tensor product. Since  $\alpha + \beta > 1$ , we can choose a  $\lambda$  such that  $\lambda < \alpha$  and  $1 - \lambda < \beta$ . For any  $a, b \in [0, T]$ , we have

$$\begin{aligned}
\left\| \int_a^b f(s) \otimes g(s) dB(s) \right\| &= \left\| \int_a^b D_{a+}^\lambda [f(s) \otimes g(s)] D_{b-}^{1-\lambda} B_{b-}(s) ds \right\| \\
&\leq \kappa \|B\|_{a,b,\beta} \int_a^b (b-s)^{\beta+\lambda-1} \|D_{a+}^\lambda [f(s) \otimes g(s)]\| ds.
\end{aligned}$$

From the definition of the Weyl derivative (2.1), we see essily

$$\begin{aligned}
\|D_{a+}^\lambda [f(s) \otimes g(s)]\| &\leq \kappa \left\{ \|f\|_{a,b} \|g\|_{a,b} (s-a)^{-\lambda} \right. \\
&\quad \left. + [\|f\|_{a,b} \|g\|_{a,b,\beta} + \|g\|_{a,b} \|f\|_{a,b,\beta}] (s-a)^{\beta-\lambda} \right\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\| \int_a^b f(s) \otimes g(s) dB(s) \right\| &\leq \kappa \|B\|_{a,b,\beta} \int_a^b (b-s)^{\beta+\lambda-1} \left\{ \|f\|_{a,b} \|g\|_{a,b} (s-a)^{-\lambda} \right. \\
&\quad \left. + [\|f\|_{a,b} \|g\|_{a,b,\beta} + \|g\|_{a,b} \|f\|_{a,b,\beta}] (s-a)^{\beta-\lambda} \right\} ds \\
&\leq \kappa \|B\|_{a,b,\beta} \left\{ \|f\|_{a,b} \|g\|_{a,b} \right. \\
&\quad \left. + [\|f\|_{a,b} \|g\|_{a,b,\beta} + \|g\|_{a,b} \|f\|_{a,b,\beta}] (b-a)^{\beta} \right\} (b-a)^{\beta}
\end{aligned}$$

which implies the lemma. ■

By a Picard iteration procedure and by the bounds that we are going to obtain, we can show that  $\mathbb{D}\Gamma(t)$  and  $\mathbb{D}z(t)$  exist under the hypothesis 5.2.

From the equation (5.4), we have

$$\begin{aligned}
\mathbb{D}\Gamma(t) &= I + \int_0^t \sigma_{xx}(s, z(s), \Gamma(s)) \mathbb{D}z(s) \otimes \int_0^s \phi(s, u) du ds \\
&\quad + \int_0^t \mathbb{D}\sigma_x(s, z(s), \Gamma(s)) \mathbb{D}\Gamma(s) \otimes \int_0^s \phi(s, u) du ds
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{D}z(t) &= \mathbb{D}\eta + \int_0^t \tilde{b}_x(s, z(s), \Gamma(s)) \mathbb{D}z(s) ds + \int_0^t \mathbb{D}\tilde{b}(s, z(s), \Gamma(s)) \mathbb{D}\Gamma(s) ds \\
&\quad + \int_0^t \sigma_x(s, z(s), \Gamma(s)) \mathbb{D}z(s) \delta B(s) \\
&\quad + \int_0^t \mathbb{D}\sigma(s, z(s), \Gamma(s)) \mathbb{D}\Gamma(s) \delta B(s) + \sigma(\cdot, z(\cdot), \Gamma(\cdot)) I_{[0,t]}(\cdot) \\
&\quad + \int_0^t \sigma_x(s, z(s), \Gamma(s)) \mathbb{D}z(s) \int_0^s \sigma_x(u, z(u), \Gamma(u)) \phi(s, u) du ds \\
&\quad + \int_0^t \mathbb{D}\sigma(s, z(s), \Gamma(s)) \mathbb{D}\Gamma(s) \int_0^s \sigma_x(u, z(u), \Gamma(u)) \phi(s, u) du ds \\
&\quad + \int_0^t \sigma(s, z(s), \Gamma(s)) \int_0^s \sigma_{xx}(u, z(u), \Gamma(u)) \mathbb{D}z(u) \phi(s, u) du ds \\
&\quad + \int_0^t \sigma(s, z(s), \Gamma(s)) \int_0^s \mathbb{D}\sigma_x(u, z(u), \Gamma(u)) \mathbb{D}\Gamma(u) \phi(s, u) du ds.
\end{aligned}$$

Similar to the lemmas 5.3 and 5.4, we can obtain

**Lemma 5.11.** *Under the hypothesis 5.2, there is a  $\kappa_B$  depending on  $\beta, p, T$  and  $\|B\|_{0,T,\beta}$  such that*

$$\left\| \frac{d}{dt} \mathbb{D}\Gamma \right\|_{0,t} \leq \kappa_B (\|\mathbb{D}z\|_{0,t} + \|\mathbb{D}\Gamma\|_{0,t}); \quad (5.48)$$

$$\begin{aligned}
\|\mathbb{D}z\|_{a,b,\beta} &\leq \kappa_B (1 + \|\mathbb{D}z\|_{0,a} + \|\mathbb{D}\Gamma\|_{0,a} \\
&\quad + \|\mathbb{D}z\|_{a,b,\beta} (b-a)^{\beta} + \|\mathbb{D}\Gamma\|_{a,b,\beta} (b-a)^{\beta}). \quad (5.49)
\end{aligned}$$

*Proof* Let us denote the integral terms in the above expression for  $\mathbb{D}z(t)$  by  $I_k$ ,  $k = 1, 2, \dots, 8$ . Let us explain how to bound  $I_3 = \int_0^t \sigma_x(s, z(s), \Gamma(s)) \mathbb{D}z(s) \delta B(s)$ .

Since  $z(s)$ ,  $\Gamma(s)$  are Hölder continuous (of exponent  $\beta$  with respect to  $s$ ),  $f(s) := \sigma_x(s, z(s), \Gamma(s))$  is then also Hölder continuous. Now the inequality (5.47) can be invoked to obtain

$$\begin{aligned} \|I_3\|_{a,b,\beta} &\leq \kappa \|B\|_{a,b,\beta} \left\{ \|f\|_{a,b} \|\mathbb{D}z\|_{a,b} \right. \\ &\quad \left. + [\|f\|_{a,b} \|\mathbb{D}z\|_{a,b,\beta} + \|\mathbb{D}z\|_{a,b} \|f\|_{a,b,\beta}] (b-a)^\beta \right\} \\ &\leq \kappa_B \left\{ \|\mathbb{D}z\|_{a,b} + [\|\mathbb{D}z\|_{a,b,\beta} + \|\mathbb{D}z\|_{a,b,\beta}] (b-a)^\beta \right\} \end{aligned} \quad (5.50)$$

since both  $\|f\|_{a,b}$  and  $\|f\|_{a,b,\beta}$  are bounded by  $\kappa_B$ . The other terms can be treated in exactly the same way. ■

**Lemma 5.12.** *Under the hypothesis, there is a  $\kappa_B$  depending on  $\beta, p, T$  and  $B_{0,T,\beta}$  such that*

$$\|\mathbb{D}\Gamma\|_{0,T} + \|\mathbb{D}z\|_{0,T} \leq \kappa_B. \quad (5.51)$$

*Proof* To show the existence of  $\mathbb{D}\Gamma(t)$  and  $\mathbb{D}z(t)$  and to show the above bound (5.51) we still use the idea in the proof of Theorem 4.1. First, we can show that the  $x_n$  defined there are  $\mathcal{H}$ -differentiable and similar bounds holds as those in Lemma 5.11 hold recursively for all  $x_n$ . This can be used to obtain the uniform bounds for all  $n$ . Since the proof is analogous that of Theorem 4.1, we shall not provide it here again. ■

**Lemma 5.13.** *Let  $G : \Omega \rightarrow \mathcal{H}$  be continuously  $\mathcal{H}$ -differentiable mapping such that*

$$\|\mathbb{D}G\|_{\mathcal{H}} \leq c < 1.$$

*Define  $\Gamma : \Omega \rightarrow \Omega$  by*

$$\Gamma(\omega) = \omega + G(\omega).$$

*Then  $\Gamma$  has a (unique) inverse  $\Lambda$  such that  $\Gamma(\Lambda(\omega)) = \Lambda(\Gamma(\omega)) = \omega$ .*

*Proof* We define

$$\Lambda_0(\omega) = \omega, \quad \Lambda_{n+1}(\omega) = \omega - G(\Lambda_n(\omega)), \quad n = 0, 1, 2, \dots$$

Then

$$\begin{aligned} \|\Lambda_{n+1}(\omega) - \Lambda_n(\omega)\|_{\mathcal{H}} &= \|G(\Lambda_n(\omega)) - G(\Lambda_{n-1}(\omega))\|_{\mathcal{H}} \\ &\leq c \|\Lambda_n(\omega) - \Lambda_{n-1}(\omega)\|_{\mathcal{H}} \leq \dots \\ &\leq c^n \|\Lambda_1(\omega) - \Lambda_0(\omega)\|_{\mathcal{H}}. \end{aligned}$$

This means that

$$\Lambda_n(\omega) - \omega = \sum_{k=1}^n (\Lambda_k(\omega) - \Lambda_{k-1}(\omega))$$

is a Cauchy sequence in  $\mathcal{H}$ . Thus  $\Lambda_n(\omega)$  converges to an element  $\Lambda(\omega)$  in  $\Omega$ . From the construction of  $\Lambda_n$  we see that  $\Lambda$  satisfies

$$\Lambda(\omega) = \omega - G(\Lambda(\omega)).$$

Thus,

$$\begin{aligned} \Gamma(\Lambda(\omega)) &= \Lambda(\omega) + G(\Lambda(\omega)) \\ &= \omega - G(\Lambda(\omega)) - G(\Lambda(\omega)) = \omega. \end{aligned}$$

This exactly means that  $\Lambda$  is the inverse of  $\Gamma$ . ■

Now we can state the main theorem of this paper.

**Theorem 5.14.** *Let  $b, \sigma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  satisfy the hypothesis (5.2). Then, the equation (5.1) has a unique solution  $x(t)$  up to some positive random time  $\tau > 0$ . This means that there is a unique positive random time  $\tau > 0$  such that*

$$x(t \wedge \tau) = \eta + \int_0^{t \wedge \tau} b(s, x(s), \omega) ds + \int_0^{t \wedge \tau} \sigma(s, x(s), \omega) dB(s), \quad \forall t \in [0, T]. \quad (5.52)$$

*Proof* According to Lemma 5.9 the remaining main task is to show the existence of an inverse  $\Lambda(t)$  of  $\Gamma(t)$ . Since the existence and uniqueness of the solution of the system (5.4) is known by Theorem 5.7, we only need to consider the first equation of the system (5.4), which is

$$\Gamma(t) = \omega + \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s)) \int_0^s \phi(s, u) du ds.$$

Due to the presence of  $z(t)$  in the above equation it is hard to prove that  $\Gamma(t)$  has an inverse for all  $t \in [0, T]$ . In fact, since  $z(t) = z(t, \omega)$  has been proved to exist, we may write the above equation as

$$\Gamma(t) = \omega + \int_0^t \hat{\sigma}(s, \omega, \Gamma(s)) ds,$$

where  $\hat{\sigma}(s, \omega, \Gamma) = \tilde{\sigma}(s, z(s, \omega), \Gamma) \int_0^s \phi(s, u) du ds$  is mapping from  $[0, T] \times \Omega \times \Omega \rightarrow \mathcal{H}$ . [As earlier we may replace  $\Gamma(t) - \omega$  by  $\Gamma(t)$  so that  $\hat{\sigma}$  is a mapping from  $[0, T] \times \Omega \times \mathcal{H} \rightarrow \mathcal{H}$ .] Or we can write the following differential equation in  $\mathcal{H}$ :

$$\dot{\Gamma}(t) = \hat{\sigma}(t, \omega, \Gamma(t)).$$

The dependence on  $\omega$  in the coefficient  $\hat{\sigma}$  may prevent the solution  $\Gamma(t)$  to have inverse for all time  $t$ . To explain we give one example in one dimension ( $\dim(\Omega) = \dim(\mathcal{H}) = 1$ ). We let  $\hat{\sigma}(t, \omega, \Gamma) = -\omega - \Gamma$ . Then the solution to the equation with initial condition  $\Gamma(0) = \omega$  is explicitly given by  $\Gamma(t, \omega) = 2\omega e^{-t} - \omega$ . But  $\Gamma(t, \omega) = 0$  when  $t = \ln 2$ . So,  $\Gamma(t, \omega)$  is not invertible when  $t = \ln 2$ . Due to the above example, we are only seeking the inverse of  $\Gamma(t)$  when  $t$  is sufficiently small (but strictly positive).

We shall prove that  $\Gamma(t) : \Omega \rightarrow \Omega$  has an inverse when  $t$  is sufficiently small. Given an arbitrarily fixed positive number  $R$ , we define the following random time:

$$\tau_R = \tau_R(\omega) = \inf \{t > 0, \quad |B(t, \omega)| > R, \|B\|_{0, t, \beta} > R\}. \quad (5.53)$$

Since we have chosen a version of the Brownian motion  $B$  such that  $B(t)$  is Hölder continuous, we see that  $0 < \tau_R < \infty$  for all  $\omega \in \Omega$ .

Now we define

$$B_R(t, \omega) = \begin{cases} B(t, \omega) & \text{when } 0 \leq t \leq \tau_R \\ B(\tau_R, \omega) & \text{when } t \geq \tau_R. \end{cases} \quad (5.54)$$

It is clear that

$$\sup_{0 \leq t \leq T} |B_R(t)| \leq R \quad \text{and} \quad \|B_R\|_{a, b, \beta} \leq 2R \quad \text{for any } 0 \leq a < b \leq T.$$



Now we consider the equation (5.4) with  $B$  replaced by  $B_R$  and the corresponding solutions are replaced by  $\Gamma_R$  and  $z_R$ :

$$\begin{cases} \Gamma_R(t) = \omega + \int_0^t \tilde{\sigma}(s, z_R(s), \Gamma_R(s)) \int_0^s \phi(s, u) du ds; \\ z_R(t) = \eta(\omega) + \int_0^t \tilde{b}(s, z_R(s), \Gamma_R(s)) ds + \int_0^t \sigma(s, z_R(s), \Gamma_R(s)) \delta B_R(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z_R(s), \Gamma_R(s)) \tilde{\sigma}(u, z_R(u), \Gamma_R(u)) \phi(s, u) du ds. \end{cases} \quad (5.55)$$

By the inequality (5.51) we see that

$$\|\mathbb{D}\Gamma_R\|_{0,T} + \|\mathbb{D}z_R\|_{0,T} \leq c_R,$$

where  $c_R$  is a constant independent of  $B$  (then independent of  $\omega$ ). This combined with Lemma 5.11 yields

$$\left\| \frac{d}{ds} \mathbb{D}\Gamma(s) \right\|_{0,T} \leq c_R. \quad (5.56)$$

On the other hand, we can write

$$\mathbb{D}\Gamma(t) = I + G(t, \omega), \quad \text{where} \quad G(t, \omega) = \int_0^t \frac{d}{ds} \mathbb{D}\Gamma(s) ds.$$

The inequality (5.56) implies that

$$\|\mathbb{D}G(t, \omega)\| \leq c_R t \leq 1/2, \quad \text{if } t \leq 1/(2c_R).$$

Thus, from Lemma 5.13 it follows that  $\Gamma(t) : \Omega \rightarrow \Omega$  has a (unique) inverse  $\Lambda(t)$  when  $t \leq t_0 := 1/(2c_R)$ . Thus, the system of equation (5.55) has a unique solution such that  $\Gamma_R(t)$  has a (unique) inverse  $\Lambda_R(t)$ . By Lemma 5.9 we see that  $x_R(t) = z_R(t, \Lambda_R(t))$  is a solution to

$$\begin{aligned} x_R(t) &= \eta + \int_0^t \tilde{b}(s, x_R(s), \omega) ds + \int_0^t \sigma(s, x_R(s), \omega) \delta B_R(s) \\ &\quad - \int_0^t \tilde{\sigma}(s, x(s), \omega) \mathbb{D}_s^\phi x_R(s) ds, \quad 0 \leq t \leq t_0. \end{aligned}$$

But when  $t \leq \tau_R$ ,  $B_R(t) = B(t)$ . Then when  $t \leq t_0 \wedge \tau_R$ , we have

$$\begin{aligned} x_R(t) &= \eta + \int_0^t \tilde{b}(s, x_R(s), \omega) ds + \int_0^t \sigma(s, x_R(s), \omega) \delta B(s) \\ &\quad - \int_0^t \tilde{\sigma}(s, x(s), \omega) \mathbb{D}_s^\phi x_R(s) ds, \quad 0 \leq t \leq t_0 \wedge \tau_R. \end{aligned}$$

This can also be written as

$$x_R(t \wedge \tau_R) = \eta + \int_0^{t \wedge \tau_R} \tilde{b}(s, x_R(s), \omega) ds + \int_0^{t \wedge \tau_R} \sigma(s, x_R(s), \omega) dB(s).$$

The theorem is then proved. ■

## 6. LINEAR AND QUASILINEAR CASES

**6.1. Quasilinear case.** If the diffusion coefficient  $\sigma$  satisfies

$$\sigma(s, x, \omega) = a_1(s, \omega)x + a_0(s, \omega), \quad (6.1)$$

then we say equation (1.2) is *quasilinear stochastic differential equation* driven by fractional Brownian motion. Associated with this equation, the corresponding system of equations (5.4) can be written as

$$\begin{cases} \Gamma(t) = \omega + \int_0^t a_1(s, \Gamma(s)) \int_0^s \phi(s, u) du ds \\ z(t) = \eta + \int_0^t \tilde{b}(s, z(s), \Gamma(s)) ds + \int_0^t \sigma(s, z(s), \Gamma(s)) \delta B(s) \\ \quad + \int_0^t \sigma(s, z(s), \Gamma(s)) \int_0^s a_1(u, \Gamma(u)) \phi(s, u) du ds, \end{cases} \quad (6.2)$$

where

$$\tilde{b}(t, x, \omega) = \tilde{b}(s, x, \omega) := b(s, x, \omega) - x \mathbb{D}_s^\phi \sigma_1(s, \omega) - \mathbb{D}_s^\phi \sigma_0(s, \omega). \quad (6.3)$$

This system is decoupled. We can first solve the above first equation.

**Proposition 6.1.** *Assume that  $a_1(t, \omega)$  is uniformly Lipschitz in  $\omega$  with respect to  $\mathcal{H}$  norm. Namely, there is a positive constant  $\kappa$  such that*

$$|a_1(t, \omega + h) - a_1(t, \omega)| \leq \kappa \|h\|_{\mathcal{H}}, \quad \forall t \in [0, T], \omega \in \Omega, h \in \mathcal{H}. \quad (6.4)$$

*Then the first equation in (6.2) has a unique solution  $\Gamma(t)$ . For all  $t \in [0, T]$ ,  $\Gamma(t) : \Omega \rightarrow \Omega$  has an inverse  $\Lambda(t)$ . Moreover, the inverse  $\Lambda(t)$  is given by  $\Lambda(t) = \Lambda(t, t)$ , where  $\{\Lambda(\cdot, t)\}$  satisfies*

$$\Lambda(s, t) = \omega + \int_0^s a_1(t - v, \Lambda(v, t)) \int_0^v \phi(v, u) du dv, \quad 0 \leq s \leq t. \quad (6.5)$$

*Proof* From the general dynamic system theory we see that for any  $t_0 \in [0, T]$ , there is a unique solution  $\Gamma(t) = \Gamma(t, t_0, \omega)$  such that the first equation of (6.2) has a unique solution for all  $t \in [0, T]$  (even when  $t < t_0$ ) such that  $\Gamma(t_0, t_0, \omega) = \omega$  and  $\Gamma(t, t_0, \omega)$  satisfies the flow property:

$$\Gamma(t, s, \Gamma(s, t_0, \omega)) = \Gamma(t + s, t_0, \omega), \quad \forall t_0, s, t \in [0, T] \text{ such that } s + t \in [0, T].$$

This can be used to show the proposition easily. ■

Once we obtain  $\Gamma(t, \omega)$  we can substitute it into the second equation in (6.2) to obtain the following equation

$$z(t) = \eta + \int_0^t \mathbf{b}(s, z(s), \omega) ds + \int_0^t \sigma(s, z(s), \Gamma(s)) \delta B(s), \quad (6.6)$$

where

$$\begin{aligned} \mathbf{b}(s, z, \omega) &= \tilde{b}(s, z, \Gamma(s)) + \sigma(s, z, \Gamma(s)) \int_0^s a_1(u, \Gamma(u)) \phi(s, u) du \\ &= b(s, x, \Gamma(s)) - x \mathbb{D}_s^\phi \sigma_1(s, \Gamma(s)) - \mathbb{D}_s^\phi \sigma_0(s, \Gamma(s)) \\ &\quad + \sigma(s, z, \Gamma(s)) \int_0^s a_1(u, \Gamma(u)) \phi(s, u) du. \end{aligned}$$

Now we can use Lemma 5.9 to obtain the following theorem.

**Theorem 6.2.** *Let the diffusion coefficient  $\sigma$  be given by (6.1). Let  $\Gamma$  be the unique solution to the first equation of (6.2). Let  $z$  be the unique solution to the second equation of (6.2). Then (1.2) has a unique solution  $x(t)$  which is given by*

$$x(t) = z(t, \Lambda(t)). \quad (6.7)$$

Moreover, there are positive constants  $c_1$  and  $c_2$ ,  $\Delta \in [0, T]$ , depending only on  $p, \beta, T$  such that for all  $0 \leq a < b \leq T$ ,  $b - a \leq \Delta$

$$\begin{cases} \sup_{0 \leq t \leq T} |x(t)| \leq c_2 \exp \left\{ c_1 \|B\|_{0,T,\beta}^{1/\beta} \right\} \\ \|x\|_{a,b,\beta} \leq c_2 \exp \left\{ c_1 \|B\|_{0,T,\beta}^{1/\beta} \right\}. \end{cases} \quad (6.8)$$

*Proof* The first inequality in (6.8) is a direct consequence of (5.35) and the above second inequality is the consequence of (5.36) together with an easy bound for  $\frac{d}{dt}\Lambda(t)$ . ■

Let us now try to solve equation (6.6), which can be written as

$$z(t) = \eta + \int_0^t \mathbf{b}(s, z(s), \omega) ds + \int_0^t [a_1(s, \Gamma(s))z(s) + a_0(s, \Gamma(s))] \delta B(s).$$

Namely,

$$z(t) - \int_0^t [a_1(s, \Gamma(s))z(s) + a_0(s, \Gamma(s))] \delta B(s) = \eta + \int_0^t \mathbf{b}(s, z(s), \omega) ds. \quad (6.9)$$

Or

$$dz(t) - [a_1(t, \Gamma(t))z(t) + a_0(t, \Gamma(t))] \delta B(s) = \mathbf{b}(t, z(t), \omega) dt. \quad (6.10)$$

Let

$$\begin{cases} A_1(t) = \exp \left\{ - \int_0^t a_1(s, \Gamma(s)) \delta B(s) \right\} \\ A_2(t) = \int_0^t A_1(s) a_0(s, \Gamma(s)) \delta B(s). \end{cases} \quad (6.11)$$

Denote

$$y(t) = A_1(t)z(t) - A_2(t). \quad (6.12)$$

Then the equation (6.9) can be written as

$$\begin{aligned} dy(t) &= A_1(t) \{ dz(t) - [a_1(t, \Gamma(t))z(t) + a_0(t, \Gamma(t))] \delta B(t) \} \\ &= A_1(t) \mathbf{b}(t, z(t), \omega) dt \\ &= \mathcal{B}(t, y(t), \omega) dt, \end{aligned} \quad (6.13)$$

where

$$\mathcal{B}(t, y) = A_1(t) \mathbf{b}(t, A_1^{-1}(t)(y + A_2(t)), \omega) \quad (6.14)$$

with  $A_1^{-1}(t) = \exp \left\{ \int_0^t a_1(r, \Gamma(r)) \delta B(r) \right\}$ . This equation is a (pathwise) ordinary differential equation and can be solved by classical method.

To summarize here is how we can solve the quasilinear equation of the following form

$$dx(t) = b(t, x(t), \omega) dt + [a_1(t, \omega)x(t) + a_0(t, \omega)] dB(t), \quad x(0) = \eta(\omega). \quad (6.15)$$

- (i) First we solve the first equation in (6.2) to obtain  $\Gamma(t)$ .
- (ii) Then we solve (6.5) to obtain the inverse  $\Lambda(t) = \Lambda(t, t)$  of  $\Gamma(t)$ .
- (iii) Define  $A_1$  and  $A_2$  by (6.11).
- (iv) Define  $\mathcal{B}(t, y) = \mathcal{B}(t, y, \omega)$  by (6.14) and solve the (ordinary differential) equation  $\dot{y}(t) = \mathcal{B}(t, y(t))$ ,  $y(0) = \eta$ .
- (v) Let  $z(t, \omega) = A_1^{-1}(t, \omega)(y(t, \omega) + A_2(t, \omega))$ .
- (vi) The solution  $x(t)$  is then given by  $x(t, \omega) = z(t, \Lambda(t, \omega))$ .

**Remark 6.3.** If  $b$  and  $\sigma$  are deterministic, then the system of equations (5.4) becomes

$$\begin{cases} \Gamma(t) = \omega + \int_0^t \tilde{\sigma}(s, z(s), \Gamma(s)) \int_0^s \phi(s, u) du ds; \\ z(t) = \eta(\omega) + \int_0^t \tilde{b}(s, z(s)) ds + \int_0^t \sigma(s, z(s)) \delta B(s) \\ \quad + \int_0^t \int_0^s \sigma(s, z(s)) \tilde{\sigma}(u, z(u)) \phi(s, u) du ds. \end{cases} \quad (6.16)$$

This system of equations is also decoupled. One may first solve the above second equation to obtain  $z(t, \omega)$  and then substitute it into the above first equation to obtain  $\Gamma(t)$ . However, the main difficulty remains to study the invertibility of  $\Gamma(t) : \Omega \rightarrow \Omega$ , which is hard as explained in the proof of Theorem 5.14.

**6.2. Linear case.** If  $\sigma$  is linear as in the previous subsection and if  $b$  is also linear, i.e.

$$\begin{cases} \sigma(s, x, \omega) = a_1(s, \omega)x + a_0(s, \omega) \\ b(s, x, \omega) = \beta_1(s, \omega)x + \beta_0(s, \omega), \end{cases} \quad (6.17)$$

then the equation is called linear equation.

$\Gamma$  and  $\Lambda$  can be found in the same way as in the quasilinear case. As we shall see that  $z$  also satisfies a linear equation, we explain how to obtain the explicit form for  $z(t)$ . First, notice that the second equation in (6.2) becomes

$$\begin{aligned} z(t) &= \eta + \int_0^t \beta_1(s, \Gamma(s)) z(s) ds + \int_0^t \beta_0(s, \Gamma(s)) ds \\ &\quad - \int_0^t [\mathbb{D}_s^\phi a_1](s, \Gamma(s)) z(s) ds - \int_0^t [\mathbb{D}_s^\phi a_0](s, \Gamma(s)) ds \\ &\quad + \int_0^t a_1(s, \Gamma(s)) z(s) \delta B(s) + \int_0^t a_0(s, \Gamma(s)) \delta B(s) \\ &\quad + \int_0^t a_1(s, \Gamma(s)) \int_0^s a_1(u, \Gamma(u)) \phi(s, u) du z(s) ds \\ &\quad + \int_0^t a_0(s, \Gamma(s)) \int_0^s a_1(u, \Gamma(u)) \phi(s, u) du ds. \end{aligned} \quad (6.18)$$

Introduce

$$\begin{aligned} \Phi(t, s) &:= \exp \left( \int_s^t [\beta_1(u, \Gamma(u)) - (\mathbb{D}_u^\phi a_1)(u, \Gamma(u))] du + \int_s^t a_1(u, \Gamma(u)) \delta B(u) \right. \\ &\quad \left. + \int_s^t a_1(u, \Gamma(u)) \int_0^u a_1(v, \Gamma(v)) \phi(u, v) dv du \right). \end{aligned} \quad (6.19)$$

Then the above equation (6.20) can be solved explicitly as

$$\begin{aligned} z(t) &= \Phi(t, 0)\eta + \int_0^t \Phi(t, s) [\beta_0(s, \Gamma(s)) - (\mathbb{D}_s^\phi a_0)(s, \Gamma(s))] ds \\ &\quad + \int_0^t \Phi(t, s) a_0(s, \Gamma(s)) \delta B(s) \\ &\quad + \int_0^t \Phi(t, s) a_0(s, \Gamma(s)) \int_0^s a_1(v, \Gamma(v)) \phi(s, v) dv ds. \end{aligned} \quad (6.20)$$

Thus the solution becomes

$$x(t) = z(t, \Lambda(t)). \quad (6.21)$$

**Example 6.4.** If  $b(s, x, \omega) = b(s)x$  and  $\sigma(s, x, \omega) = a(s)x$  and  $\eta = x_0$ , where  $b(s)$  and  $a(s)$  are deterministic function of  $s$ , then the first equation of (6.2) becomes

$$\Gamma(t) = \omega + \int_0^t a(s) \int_0^s \phi(s, u) du ds.$$

Thus

$$\Lambda(t) = \omega - \int_0^t h(t, u) du, \quad (6.22)$$

where

$$h(t, u) = \int_0^t a(s) \phi(s, u) ds. \quad (6.23)$$

Since  $\mathbb{D}^\phi a = 0$ , we have

$$\Phi(t, s) = \exp \left\{ \int_s^t b(u) du + \int_s^t a(u) \delta B(u) + \int_s^t a(u) \int_0^u a(v) \phi(u, v) dv du \right\}. \quad (6.24)$$

Now since  $\mathbb{D}^\phi b = 0$  and  $\beta_0 = a_0 = 0$ , we have

$$z(t) = \Phi(t, 0)x_0.$$

Since  $\Phi(t, s) = \Phi(t, s, \omega)$  still depends on  $\omega$ . Using lemma 3.1, we have

$$\begin{aligned} \Psi(t, s) &:= \Phi(t, s, \Lambda(t)) \\ &= \exp \left\{ \int_s^t b(u) du + \int_s^t a(u) \delta B(u) - \int_s^t du \int_u^t a(u) a(s) \phi(s, u) ds \right\} \end{aligned} \quad (6.25)$$

Thus the solution to

$$dx(t) = b(t)x(t)dt + a(t)x(t)dB(t) \quad (6.26)$$

is given by

$$\begin{aligned} x(t) &= \Psi(t, 0)x_0 \\ &= \exp \left\{ \int_0^t b(u) du + \int_0^t a(u) \delta B(u) \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \int_0^t a(u) a(s) \phi(s, u) du ds \right\} x_0. \end{aligned} \quad (6.27)$$

This is well-known (see for example [1, 4, 6, 9, 13]).

**Example 6.5.** In the same way as above example, we can solve the following linear stochastic differential equation

$$dx(t) = [b(t)x(t) + \beta(t)]dt + [a(t)x(t) + \alpha(t)]dB(t), \quad x(0) = x_0, \quad (6.28)$$

where  $x_0 \in \mathbb{R}$ ,  $b(t), \beta(t), a(t), \alpha(t)$  are deterministic functions, to obtain

$$\begin{aligned} x(t) &= \Psi(t, 0)x_0 + \int_0^t \Psi(t, s)\beta(s)ds + \int_0^t \Psi(t, s)\alpha(s)\delta B(s) \\ &\quad + \int_0^t \Psi(t, s)\alpha(s) \int_0^s a(v)\phi(s, v)dv ds, \end{aligned} \quad (6.29)$$

where  $\Psi$  is given by (6.25).

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